ARMY RESEARCH LABORATORY

Detection of a Ferromagnetic Microwire

by Frank J. Crowne

ARL-TR-6115 September 2012

NOTICES

Disclaimers

Army Research Laboratory
Adelphi, MD 20783-1197

ARL-TR-6115 September 2012

Detection of a Ferromagnetic Microwire

Frank J. Crowne
Sensors and Electron Devices Directorate, ARL

REPORT DOCUMENTATION PAGE						*Form Approved OMB No. 0704-0188*
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. **PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.**						
1. REPORT DATE *(DD-MM-YYYY)* September 2012		**2. REPORT TYPE** Final				**3. DATES COVERED (From - To)**
4. TITLE AND SUBTITLE Detection of a Ferromagnetic Microwire						**5a. CONTRACT NUMBER**
						5b. GRANT NUMBER
						5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Frank J. Crowne						**5d. PROJECT NUMBER**
						5e. TASK NUMBER
						5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) U.S. Army Research Laboratory ATTN: RDRL-SER-E 2800 Powder Mill Road Adelphi, MD 20783-1197						**8. PERFORMING ORGANIZATION REPORT NUMBER** ARL-TR-6115
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)						**10. SPONSOR/MONITOR'S ACRONYM(S)**
						11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.						
13. SUPPLEMENTARY NOTES						
14. ABSTRACT This report, which supplements the material presented in the *IEEE Transactions in Magnetics* publication entitled "Two-Frequency Excitation of a Ferromagnetic Microwire," describes a simple configuration of antenna beams, polarizations, and wire placement calculations, which are used in calculating the response of a magnetic wire to ferromagnetic resonance (FMR) excitation by an external magnetic field.						
15. SUBJECT TERMS Ferromagnetic resonance, nonlinear response, magnetic microwires						
16. SECURITY CLASSIFICATION OF:				**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON** Frank J. Crowne
a. REPORT Unclassified	**b. ABSTRACT** Unclassified	**c. THIS PAGE** Unclassified		UU	28	**19b. TELEPHONE NUMBER** (*Include area code*) (301) 394-5759

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

Contents

INTENTIONALLY LEFT BLANK.

1. Introduction

As described in the publication “Two-Frequency Excitation of a Ferromagnetic Microwire” (*1*), the remote sensing of ferroelectric microwires based on ferromagnetic resonance (FMR) has been proposed as a way to passively collect information from a novel type of “radio frequency identification (RFID) card” that can be attached to a target under surveillance (*2*). Implementation of this proposal requires a careful study of the external scattering problem of electromagnetic (EM) waves from the microwire, which is the topic of this technical report. A detailed description of the physics of ferromagnetic microwires can be found in reference 1 and ARL-TR-6114 (*3*).

2. The Wire Problem

Assume that a ferroelectric microwire located at the coordinate origin and oriented vertical to the ground, i.e., along the “*z*-axis”, is illuminated by two external magnetic fields, which we refer to as “signal” and “pump” fields, both of which are the magnetic components of linearly polarized EM beams. The “illumination” geometries correspond to radar applications: if both beams originate from the same antenna, we speak of a “monostatic” geometry, while for signals originating from different antennas the geometry is referred to as “bistatic.” In this report, we discuss the monostatic scenario only, i.e., we assume that both beams originate from the same antenna and propagate horizontally and parallel to one another along, e.g., the *x*-axis. The geometry is shown in figure 1.

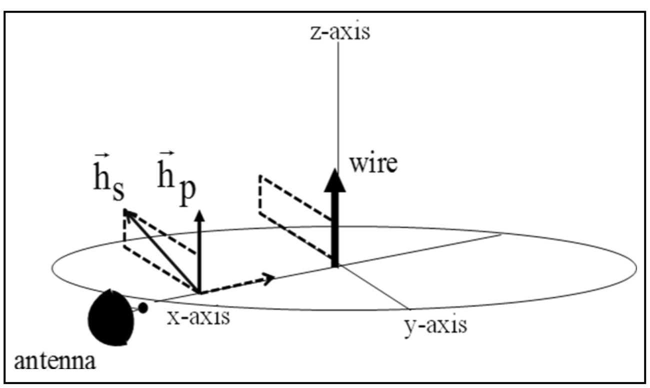

Figure 1 Monostatic geometry for wire illumination—propagation vector.

The two fields are characterized by both spatial and temporal phase information, i.e., if $\left(\mathcal{H}_{\mathrm{p}x}, \mathcal{H}_{\mathrm{p}y}, \mathcal{H}_{\mathrm{pz}}\right)$ and $\left(\mathcal{H}_{\mathrm{s}x}, \mathcal{H}_{\mathrm{s}y}, \mathcal{H}_{\mathrm{sz}}\right)$ are real magnitudes of the (strong) pump field and the (weak) signal field, the amplitudes of the rectangular components of the fields are given by $\mathrm{h}_{\mathrm{p}x}(t) = \mathrm{Re}\left(\mathcal{H}_{\mathrm{p}x} e^{-i\left(\omega t + \theta_{\mathrm{p}x}\right)}\right)$, $\mathrm{h}_{\mathrm{p}y}(t) = \mathrm{Re}\left(\mathcal{H}_{\mathrm{p}y} e^{-i\left(\omega t + \theta_{\mathrm{p}y}\right)}\right)$, and $\mathrm{h}_{\mathrm{pz}}(t) = \mathrm{Re}\left(\mathcal{H}_{\mathrm{pz}} e^{-i\left(\omega t + \theta_{\mathrm{pz}}\right)}\right)$, where spatial phase information associated with the wave polarization is included by letting $\theta_{\mathrm{p}x} \neq \theta_{\mathrm{p}y} \neq \theta_{\mathrm{pz}}$. For our case, we assume that the pump field at the antenna is linearly polarized along the z-axis, so that $\mathrm{h}_{\mathrm{p}x} = 0$, $\mathrm{h}_{\mathrm{p}y} = 0$, and $\mathrm{h}_{\mathrm{pz}} = \mathcal{H}_{\mathrm{p}c}$, where $\mathcal{H}_{\mathrm{p}c}$ is the total pump-field magnitude. In the same way, the (weak) signal field at the antenna is linearly polarized in the yz plane, with its electric field vector making an angle φ with the z-axis, having components $\mathrm{h}_{sx} = 0$, $\mathrm{h}_{sy} = \mathcal{H}_{\mathrm{s}c}\cos\varphi$, and $\mathrm{h}_{sz} = \mathcal{H}_{\mathrm{s}c}\sin\varphi$, where $\mathcal{H}_{\mathrm{s}c}$ is the total signal-field magnitude. Then, the wave components at the target are

$$\mathrm{h}_{\mathrm{p}}(t) = \mathrm{Re}\left[\begin{pmatrix} 0 \\ 0 \\ \mathcal{H}_{\mathrm{p}c} \end{pmatrix} e^{-i\left(\beta x + \omega t\right)}\right] \qquad \mathrm{h}_{\mathrm{s}}(t) = \mathrm{Re}\left[\begin{pmatrix} 0 \\ \mathcal{H}_{\mathrm{s}c}\cos\varphi \\ \mathcal{H}_{\mathrm{s}c}\sin\varphi \end{pmatrix} e^{-i\left(\beta x + \omega t\right)}\right] \tag{1}$$

Following reference 1, we introduce the following complex fields (italics)

$$\begin{aligned} h_{\mathrm{p}x} &= 0 & h_{\mathrm{s}x} &= 0 \\ h_{\mathrm{p}y} &= 0 & h_{\mathrm{s}y} &= \mathcal{H}_{\mathrm{s}c}\cos\varphi \\ h_{\mathrm{pz}} &= \mathcal{H}_{\mathrm{p}c} & h_{\mathrm{sz}} &= \mathcal{H}_{\mathrm{s}c}\sin\varphi \end{aligned} \tag{2}$$

and the circular field components, defined as follows:

$$h_{px} + ih_{py} = \mathcal{H}_{px} e^{-i\theta_{px}} + i\mathcal{H}_{py} e^{-i\theta_{py}}, \; h_{pz} = \mathcal{H}_{pz} e^{-i\theta_{sz}}$$
$$\mathcal{H}_{px} = 0, \; \mathcal{H}_{py} = 0, \; \mathcal{H}_{pz} = \mathcal{H}_{pc}, \; \theta_{pz} = 0$$
$$h_{sx} + ih_{sy} = \mathcal{H}_{sx} e^{-i\theta_{sx}} + i\mathcal{H}_{sy} e^{-i\theta_{sy}}, \; h_{sz} = \mathcal{H}_{sz} e^{-i\theta_{sz}}$$
$$\mathcal{H}_{sx} = 0, \; \mathcal{H}_{sy} = \mathcal{H}_{sc}\cos\varphi, \; \mathcal{H}_{sz} = \mathcal{H}_{sc}\sin\varphi, \; \theta_y = 0, \; \theta_z = 0$$
$$\Rightarrow \begin{cases} h_{px} + ih_{py} = 0 \\ h_{sx} + ih_{sy} = i\mathcal{H}_{sc}\cos\varphi \\ \left(h_{px} - ih_{py}\right)^* = 0 \\ \left(h_{sx} - ih_{sy}\right)^* = i\mathcal{H}_{sc}\cos\varphi \end{cases} \tag{3}$$

In order to estimate the power radiated by FMR excitation of the microwire, it is necessary first to describe the dynamics of its magnetization in the presence of an external time-dependent magnetic field. These dynamics are governed by the Landau-Lifshits-Gilbert (LLG) equation (*4*):

$$\frac{\partial \vec{M}}{\partial t} = \gamma \vec{M} \times \vec{H}_{eff}(t) - \frac{\alpha}{\left|\vec{M}_{st}\right|} \vec{M} \times \frac{\partial \vec{M}}{\partial t} \tag{4}$$

The material parameters of the wire include $\vec{M}_{st}$, the static magnetization of the material; γ, the gyromagnetic ratio; and α, the damping constant. $\vec{H}_{eff}(t)$ is an effective magnetic field, which includes the DC anisotropy field $\vec{H}_a$ that fixes the direction of $\vec{M}_{st}$ in the material, the depolarization field (*5*) due to the body's shape, and the external time-dependent applied field. Note that in the absence of an ac magnetic field, there is no depolarization field, and so $\vec{M}_{st} \times \vec{H}_a = 0$ under DC conditions, i.e., there is no DC torque. The first term is the torque exerted by the effective magnetic field $\vec{H}_{eff}(t)$, while the second term gives rise to damping via eddy currents (*6*). The form of this equation implies that $\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0 \Rightarrow \left|\vec{M}\right|^2$ is constant in time. The RF magnetic field $\vec{\mathrm{h}}(\mathrm{t})$ generates an RF magnetization $\vec{\mathrm{m}}$ that under remote-sensing conditions is small compared to $\vec{M}_{st}$. The dynamics of this magnetization, in particular its decay with time, is strongly affected by the conservation of length of the total magnetization vector, whose endpoint is constrained to lie on a spherical surface at all times.

Let us further specify the perturbed magnetization and effective magnetic field in the LLG equation as follows: $\vec{M} = \vec{M}_{st} + \vec{\mathrm{m}}$, $\vec{H}_{eff} = \vec{H}_a + \vec{\mathrm{h}} - \mathrm{N}\vec{\mathrm{m}}$, where $\vec{M}_{st} \times \vec{H}_a = 0$ and

$$
\mathrm{N} = \begin{pmatrix} \tilde{N} & 0 & 0 \\ 0 & \tilde{N} & 0 \\ 0 & 0 & N_z \end{pmatrix} \tag{5}
$$

is the depolarization tensor; for a long wire, $\tilde{N} = 1/2$ and $N_z \approx 0$. As described in reference 1, we can introduce a scaled dimensionless magnetization $\vec{\mathrm{g}}$ such that $\vec{\mathrm{m}} = \left|\vec{M}_{st}\right| \vec{\mathrm{g}}$. In general, this magnetization has three components $\mathrm{g_x}, \mathrm{g_y}, \mathrm{g_z}$, where $\mathrm{g_z}$ is the component of $\vec{\mathrm{g}}$ parallel to the wire and $\left(\mathrm{g_x}, \mathrm{g_y}\right)$ is a two-dimensional vector perpendicular to the wire, with the latter small compared to the former. As shown in reference 1, under weak-signal conditions the perpendicular components of the dimensionless magnetization can be expressed in terms of a single complex number $\mathcal{G}$ at each illuminating frequency, expressions for which are

$$
\begin{cases}
\mathcal{G}_{1\mathrm{s}} = \dfrac{\gamma\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right)}{\omega_B - \omega_\mathrm{s} - i\alpha\omega_\mathrm{s}} e^{-i\omega_\mathrm{s} t} + \dfrac{\gamma\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right)^*}{\omega_B + \omega_\mathrm{s} + i\alpha\omega_\mathrm{s}} e^{i\omega_\mathrm{s} t} \\
\mathcal{G}_{1\mathrm{p}} = \dfrac{\gamma\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)_+}{\omega_B - \omega_\mathrm{p} - i\alpha\omega_\mathrm{p}} e^{-i\omega_\mathrm{p} t} + \dfrac{\gamma\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)^*}{\omega_B + \omega_\mathrm{p} + i\alpha\omega_\mathrm{p}} e^{i\omega_\mathrm{p} t}
\end{cases} \tag{6}
$$

In the geometry we have chosen, these expressions become

$$
\begin{gathered}
\mathcal{G}_{1,\mathrm{s}} = \frac{\gamma}{\omega_B} \mathcal{H}_{sc} \cos\varphi \left[\frac{i}{1 - \xi_\mathrm{s} - i\alpha\xi_\mathrm{s}} e^{-i\omega_\mathrm{s} t} + \frac{i}{1 + \xi_\mathrm{s} + i\alpha\xi_\mathrm{s}} e^{i\omega_\mathrm{s} t} \right] \\
\mathcal{G}_{1,\mathrm{p}} = 0
\end{gathered} \tag{7}
$$

while

$\mathrm{g}_{z1,\mathrm{s}} = 0$

$\mathrm{g}_{z1,\mathrm{p}} = 0$

---

3. Nonlinear Response: Second Order

From reference 1, we find that, to second order, the wire response functions satisfy the equations

$$
\begin{aligned}
(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_2 + \omega_B \mathcal{G}_2 &= -\gamma \mathrm{h}_z \mathcal{G}_1 \\
(\alpha + i)\frac{\partial}{\partial t}\mathcal{G}_2^{\,*} + \omega_B \mathcal{G}_2^{\,*} &= -\gamma \mathrm{h}_z \mathcal{G}_1^{\,*}
\end{aligned}
\tag{8}
$$

where the “sources” appearing on the right side of these equations are independent of $\mathcal{G}_2$. Introducing the notation

$$
\vec{H}_{\mathrm{s,p}} = \begin{pmatrix} \mathrm{h}_{\mathrm{s,p}x} + i\mathrm{h}_{\mathrm{s,p}y} \\ \mathrm{h}_{\mathrm{s,p}x} - i\mathrm{h}_{\mathrm{s,p}y} \end{pmatrix} = \begin{pmatrix} H_{\mathrm{s,p}} \\ H_{\mathrm{s,p}}^{\,*} \end{pmatrix}
\tag{9}
$$

and

$$
\begin{aligned}
\Gamma_{\mathrm{s,p+}} &= \frac{\gamma}{2}\frac{1}{\omega_B - \omega_{\mathrm{s,p}} - i\omega\alpha} \qquad & \Gamma_{\mathrm{s,p-}} &= \frac{\gamma}{2}\frac{1}{\omega_B + \omega_{\mathrm{s,p}} - i\omega\alpha} \\
\Gamma_{\mathrm{s,p+}}^{\,*} &= \frac{\gamma}{2}\frac{1}{\omega_B - \omega_{\mathrm{s,p}} + i\omega\alpha} \qquad & \Gamma_{\mathrm{s,p-}}^{\,*} &= \frac{\gamma}{2}\frac{1}{\omega_B + \omega_{\mathrm{s,p}} + i\omega\alpha}
\end{aligned}
\tag{10}
$$

we obtain the following expressions for the source terms:

$$
-\gamma \mathrm{h}_z \mathcal{G}_1 = -\frac{\gamma}{2}\left[\begin{array}{l}
\left(h_{\mathrm{sz}}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}}\right)e^{-2i\omega_{\mathrm{s}}t} + \left(h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{s-}}^{\,*}H_{\mathrm{s-}}^{\,*}\right)e^{+2i\omega_{\mathrm{s}}t} \\
+\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}}\right)e^{-i\omega_{\mathrm{s+p}}t} + \left(h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{p-}}^{\,*}H_{\mathrm{p-}}^{\,*} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{s-}}^{\,*}H_{\mathrm{s-}}^{\,*}\right)e^{+i\omega_{\mathrm{s+p}}t} \\
+h_{\mathrm{sz}}\Gamma_{\mathrm{s-}}^{\,*}H_{\mathrm{s-}}^{\,*} + h_{\mathrm{pz}}\Gamma_{\mathrm{p-}}^{\,*}H_{\mathrm{p-}}^{\,*} + h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}} \\
+\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p-}}^{\,*}H_{\mathrm{p-}}^{\,*} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}}\right)e^{-i\omega_{\mathrm{s-p}}t} + \left(h_{\mathrm{pz}}\Gamma_{\mathrm{s-}}^{\,*}H_{\mathrm{s-}}^{\,*} + h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}}\right)e^{+i\omega_{\mathrm{s-p}}t} \\
+h_{\mathrm{pz}}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}}e^{-2i\omega_{\mathrm{p}}t} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{p-}}^{\,*}H_{\mathrm{p-}}^{\,*}e^{+2i\omega_{\mathrm{p}}t}
\end{array}\right]
$$

$$
-\gamma \mathrm{h}_z \mathcal{G}_1^{\,*} = -\frac{\gamma}{2}\left[\begin{array}{l}
\left(h_{\mathrm{sz}}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}}\right)e^{-2i\omega_{\mathrm{s}}t} + \left(h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{s+}}^{\,*}H_{\mathrm{s+}}^{\,*}\right)e^{+2i\omega_{\mathrm{s}}t} + \\
+\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}}\right)e^{-i\omega_{\mathrm{s+p}}t} + \left(h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{p+}}^{\,*}H_{\mathrm{p+}}^{\,*} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{s+}}^{\,*}H_{\mathrm{s+}}^{\,*}\right)e^{+i\omega_{\mathrm{s+p}}t} \\
+h_{\mathrm{sz}}\Gamma_{\mathrm{s+}}^{\,*}H_{\mathrm{s+}}^{\,*} + h_{\mathrm{pz}}\Gamma_{\mathrm{p+}}^{\,*}H_{\mathrm{p+}}^{\,*} + h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}} \\
+\left(h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}} + h_{\mathrm{sz}}\Gamma_{\mathrm{p+}}^{\,*}H_{\mathrm{p+}}^{\,*}\right)e^{-i\omega_{\mathrm{s-p}}t} + \left(h_{\mathrm{sz}}^{\,*}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s+}}^{\,*}H_{\mathrm{s+}}^{\,*}\right)e^{+i\omega_{\mathrm{s-p}}t} \\
+h_{\mathrm{pz}}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}}e^{-2i\omega_{\mathrm{p}}t} + h_{\mathrm{pz}}^{\,*}\Gamma_{\mathrm{p+}}^{\,*}H_{\mathrm{p+}}^{\,*}e^{+2i\omega_{\mathrm{p}}t}
\end{array}\right]
\tag{11}
$$

Let us write the second-order response in the following spectral form:

$$\mathcal{G}_2 = \mathcal{G}_{2,2\mathrm{s}} + \mathcal{G}_{2,2\mathrm{p}} + \mathcal{G}_{2,\mathrm{s+p}} + \mathcal{G}_{2,\mathrm{s-p}} + \mathcal{G}_{2,0}\,. \tag{12}$$

Here $\mathcal{G}_{2,2\mathrm{s}}$ is the response at the 2nd harmonic of the signal field, $\mathcal{G}_{2,2\mathrm{s}}$ is the response at the 2nd harmonic of the pump field, $\mathcal{G}_{2,\mathrm{s+p}}$ is the response at the sum of the signal and pump frequencies, $\mathcal{G}_{2,\mathrm{s-p}}$ is the response at the difference of the signal and pump frequencies, and $\mathcal{G}_{2,0}$ is a DC response arising from rectification of the pump and signal. We introduce the following convention: if *A* is a sum of multi-frequency terms, we denote the sum of all the terms in *A* at the same frequency *q* by $[A]_q$.

The response $\mathcal{G}_{2,2\mathrm{s}}$ at the 2nd harmonic frequency 2s is computed as follows:

$$\begin{aligned}
&(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_{2,2\mathrm{s}} + \omega_B \mathcal{G}_{2,2\mathrm{s}} = \left[-\gamma \mathrm{h}_z \mathcal{G}_1\right]_{2\mathrm{s}} \\
&= -\frac{\gamma}{2}\left[\left(h_{\mathrm{s}z}\Gamma_{\mathrm{s}+}H_{\mathrm{s}+}\right)e^{-2i\omega_\mathrm{s}t} + \left(h_{\mathrm{s}z}\Gamma_{\mathrm{s}-}H_{\mathrm{s}-}\right)^* e^{+2i\omega_\mathrm{s}t}\right] \square -\frac{\gamma}{2}\left[S_{2,2\mathrm{s}+}e^{-2i\omega_\mathrm{s}t} + S_{2,2\mathrm{s}-}{}^* e^{+2i\omega_\mathrm{s}t}\right] \\
&(\alpha + i)\frac{\partial}{\partial t}\mathcal{G}_{2,2\mathrm{s}}{}^* + \omega_B \mathcal{G}_{2,2\mathrm{s}}{}^* = \left[-\gamma \mathrm{h}_z \mathcal{G}_1{}^*\right]_{2\mathrm{s}} \\
&= -\frac{\gamma}{2}\left[\left(h_{\mathrm{s}z}\Gamma_{\mathrm{s}-}H_{\mathrm{s}-}\right)e^{+2i\omega_\mathrm{s}t} + \left(h_{\mathrm{s}z}\Gamma_{\mathrm{s}+}H_{\mathrm{s}+}\right)^* e^{-2i\omega_\mathrm{s}t}\right] \square -\frac{\gamma}{2}\left[S_{2,2\mathrm{s}-}e^{-2i\omega_\mathrm{s}t} + S_{2,2\mathrm{s}+}{}^* e^{+2i\omega_\mathrm{s}t}\right]
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\Rightarrow \mathcal{G}_{2,2\mathrm{s}} = -\frac{\gamma}{2}\left[\frac{S_{2,2\mathrm{s}+}}{\omega_B - \omega_{2\mathrm{s}} - i\alpha\omega_{2\mathrm{s}}}e^{-2i\omega_\mathrm{s}t} + \frac{S_{2,2\mathrm{s}-}{}^*}{\omega_B + \omega_{2\mathrm{s}} + i\alpha\omega_{2\mathrm{s}}}e^{+2i\omega_\mathrm{s}t}\right] \\
&= -\frac{\gamma}{2}\left[\frac{h_{\mathrm{s}z}\Gamma_{\mathrm{s}+}H_{\mathrm{s}+}}{\omega_B - \omega_{2\mathrm{s}} - i\alpha\omega_{2\mathrm{s}}}e^{-2i\omega_\mathrm{s}t} + \frac{\left(h_{\mathrm{s}z}\Gamma_{\mathrm{s}-}H_{\mathrm{s}-}\right)^*}{\omega_B + \omega_{2\mathrm{s}} + i\alpha\omega_{2\mathrm{s}}}e^{+2i\omega_\mathrm{s}t}\right] \\
&= -\frac{\gamma}{2}\left[\frac{\Gamma_{\mathrm{s}+}}{\omega_B - \omega_{2\mathrm{s}} - i\alpha\omega_{2\mathrm{s}}}\,h_{\mathrm{s}z}H_{\mathrm{s}+}e^{-2i\omega_\mathrm{s}t} + \frac{\Gamma_{\mathrm{s}-}{}^*}{\omega_B + \omega_{2\mathrm{s}} + i\alpha\omega_{2\mathrm{s}}}\,h_{\mathrm{s}z}{}^* H_{\mathrm{s}-}{}^* e^{+2i\omega_\mathrm{s}t}\right] \\
&= -\left(\frac{\gamma}{2}\right)^2\left[\frac{h_{\mathrm{s}z}H_{\mathrm{s}+}}{\left(\omega_B - \omega_{2\mathrm{s}} - i\alpha\omega_{2\mathrm{s}}\right)\left(\omega_B - \omega_\mathrm{s} - i\alpha\omega_\mathrm{s}\right)}e^{-2i\omega_\mathrm{s}t} + \frac{h_{\mathrm{s}z}{}^* H_{\mathrm{s}-}{}^*}{\left(\omega_B + \omega_{2\mathrm{s}} + i\alpha\omega_{2\mathrm{s}}\right)\left(\omega_B + \omega_\mathrm{s} + i\alpha\omega_\mathrm{s}\right)}e^{+2i\omega_\mathrm{s}t}\right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\frac{h_{\mathrm{s}z}\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right)}{\left(1 - \xi_{2\mathrm{s}} - i\alpha\xi_{2\mathrm{s}}\right)\left(1 - \xi_\mathrm{s} - i\alpha\xi_\mathrm{s}\right)}e^{-2i\omega_\mathrm{s}t} + \frac{h_{\mathrm{s}z}{}^*\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right)^*}{\left(1 + \xi_{2\mathrm{s}} + i\alpha\xi_{2\mathrm{s}}\right)\left(1 + \xi_\mathrm{s} + i\alpha\xi_\mathrm{s}\right)}e^{+2i\omega_\mathrm{s}t}\right]
\end{aligned} \tag{14}$$

where we introduce the dimensionless frequencies $\xi_i = \dfrac{\omega_i}{\omega_B}$. Likewise, the response $\mathcal{G}_{2,2\mathrm{p}}$ at the 2nd harmonic frequency 2p is

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_{2,2\mathrm{p}} + \omega_B \mathcal{G}_{2,2\mathrm{p}} = \left[-\gamma \mathrm{h}_z \mathcal{G}_1\right]_{2\mathrm{p}}$$

$$= -\frac{\gamma}{2}\left[\left(h_{\mathrm{pz}}\Gamma_{\mathrm{p}+}H_{\mathrm{p}+}\right)e^{-2i\omega_{\mathrm{p}}t} + \left(h_{\mathrm{pz}}\Gamma_{\mathrm{p}-}H_{\mathrm{p}-}\right)^* e^{+2i\omega_{\mathrm{p}}t}\right] \square -\frac{\gamma}{2}\left[S_{2,2\mathrm{p}+}e^{-2i\omega_{\mathrm{p}}t} + S_{2,2\mathrm{p}-}{}^* e^{+2i\omega_{\mathrm{p}}t}\right] \tag{15}$$

$$(\alpha + i)\frac{\partial}{\partial t}\mathcal{G}_{2,2\mathrm{p}}{}^* + \omega_B \mathcal{G}_{2,2\mathrm{p}}{}^* = \left[-\gamma \mathrm{h}_z \mathcal{G}_1{}^*\right]_{2\mathrm{p}}$$

$$= -\frac{\gamma}{2}\left[\left(h_{\mathrm{pz}}\Gamma_{\mathrm{p}-}H_{\mathrm{p}-}\right)e^{-2i\omega_{\mathrm{p}}t} + \left(h_{\mathrm{pz}}\Gamma_{\mathrm{p}+}H_{\mathrm{p}+}\right)^* e^{+2i\omega_{\mathrm{p}}t}\right] \square -\frac{\gamma}{2}\left[S_{2,2\mathrm{p}-}e^{-2i\omega_{\mathrm{p}}t} + S_{2,2\mathrm{p}+}{}^* e^{+2i\omega_{\mathrm{p}}t}\right]$$

$$\Rightarrow \mathcal{G}_{2,2\mathrm{p}} = -\frac{\gamma}{2}\left[\frac{S_{2,2\mathrm{p}+}}{\omega_B - \omega_{2\mathrm{p}} - i\alpha\omega_{2\mathrm{p}}}e^{-2i\omega_{\mathrm{p}}t} + \frac{S_{2,2\mathrm{p}-}{}^*}{\omega_B + \omega_{2\mathrm{p}} + i\alpha\omega_{2\mathrm{p}}}e^{+2i\omega_{\mathrm{p}}t}\right]$$

$$= -\frac{\gamma}{2}\left[\frac{h_{\mathrm{pz}}\Gamma_{\mathrm{p}+}H_{\mathrm{p}+}}{\omega_B - \omega_{2\mathrm{p}} - i\alpha\omega_{2\mathrm{p}}}e^{-2i\omega_{\mathrm{p}}t} + \frac{\left(h_{\mathrm{pz}}\Gamma_{\mathrm{p}-}H_{\mathrm{p}-}\right)^*}{\omega_B + \omega_{2\mathrm{p}} + i\alpha\omega_{2\mathrm{p}}}e^{+2i\omega_{\mathrm{p}}t}\right]$$

$$= -\frac{\gamma}{2}\left[\frac{\Gamma_{\mathrm{p}+}}{\omega_B - \omega_{2\mathrm{p}} - i\alpha\omega_{2\mathrm{p}}}h_{\mathrm{pz}}H_{\mathrm{p}+}e^{-2i\omega_{\mathrm{p}}t} + \frac{\Gamma_{\mathrm{p}-}{}^*}{\omega_B + \omega_{2\mathrm{p}} + i\alpha\omega_{2\mathrm{p}}}h_{\mathrm{pz}}{}^* H_{\mathrm{p}-}{}^* e^{+2i\omega_{\mathrm{p}}t}\right]$$

$$= -\left(\frac{\gamma}{2}\right)^2\left[\frac{h_{\mathrm{pz}}H_{\mathrm{p}+}}{\left(\omega_B - \omega_{2\mathrm{p}} - i\alpha\omega_{2\mathrm{p}}\right)\left(\omega_B - \omega_{\mathrm{p}} - i\alpha\omega_{\mathrm{p}}\right)}e^{-2i\omega_{\mathrm{p}}t} + \frac{h_{\mathrm{pz}}{}^* H_{\mathrm{p}-}{}^*}{\left(\omega_B + \omega_{2\mathrm{p}} + i\alpha\omega_{2\mathrm{p}}\right)\left(\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}\right)}e^{+2i\omega_{\mathrm{p}}t}\right]$$

$$= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\frac{h_{\mathrm{pz}}\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)}{\left(1 - \xi_{2\mathrm{s}} - i\alpha\xi_{2\mathrm{s}}\right)\left(1 - \xi_{\mathrm{p}} - i\alpha\xi_{\mathrm{p}}\right)}e^{-2i\omega_{\mathrm{p}}t} + \frac{h_{\mathrm{pz}}{}^*\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)^*}{\left(1 + \xi_{2\mathrm{p}} + i\alpha\xi_{2\mathrm{p}}\right)\left(1 + \xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)}e^{+2i\omega_{\mathrm{p}}t}\right] \tag{16}$$

The DC response $\mathcal{G}_{2,0}$ is

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_{2,0} + \omega_B \mathcal{G}_{2,0} = \left[-\gamma \mathrm{h}_z \mathcal{G}_1\right]_0$$

$$\left[-\gamma \mathrm{h}_z \mathcal{G}_1\right]_0 = -\frac{\gamma}{2}\left[h_{\mathrm{sz}} \Gamma_{\mathrm{s-}}{}^{*} H_{\mathrm{s-}}{}^{*} + h_{\mathrm{pz}} \Gamma_{\mathrm{p-}}{}^{*} H_{\mathrm{p-}}{}^{*} + h_{\mathrm{sz}}{}^{*} \Gamma_{\mathrm{s+}} H_{\mathrm{s+}} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{p+}} H_{\mathrm{p+}} \right]$$

$$= -\frac{\gamma}{2}\left[\begin{array}{l} h_{\mathrm{sz}} \frac{\gamma}{2} \frac{1}{\omega_B + \omega_{\mathrm{s}} + i\alpha\omega_{\mathrm{s}}} H_{\mathrm{s-}}{}^{*} + h_{\mathrm{pz}} \frac{\gamma}{2} \frac{1}{\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}} H_{\mathrm{p-}}{}^{*} \\ + h_{\mathrm{sz}}{}^{*} \frac{\gamma}{2} \frac{1}{\omega_B - \omega_{\mathrm{s}} - i\alpha\omega_{\mathrm{s}}} H_{\mathrm{s+}} + h_{\mathrm{pz}}{}^{*} \frac{\gamma}{2} \frac{1}{\omega_B - \omega_{\mathrm{p}} - i\alpha\omega_{\mathrm{p}}} H_{\mathrm{p+}} \end{array}\right]$$

$$\Rightarrow \mathcal{G}_{2,0} = -\left(\frac{\gamma}{2\omega_B}\right)\left(\frac{\gamma}{2}\right)\left[\begin{array}{l} h_{\mathrm{sz}} \frac{1}{\omega_B + \omega_{\mathrm{s}} + i\alpha\omega_{\mathrm{s}}} H_{\mathrm{s-}}{}^{*} + h_{\mathrm{sz}}{}^{*} \frac{1}{\omega_B - \omega_{\mathrm{s}} - i\alpha\omega_{\mathrm{s}}} H_{\mathrm{s+}} \\ + h_{\mathrm{pz}} \frac{1}{\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}} H_{\mathrm{p-}}{}^{*} + h_{\mathrm{pz}}{}^{*} \frac{1}{\omega_B - \omega_{\mathrm{p}} - i\alpha\omega_{\mathrm{p}}} H_{\mathrm{p+}} \end{array}\right]$$

$$= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\begin{array}{l} h_{\mathrm{sz}} \frac{1}{1 + \xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}} H_{\mathrm{s-}}{}^{*} + h_{\mathrm{sz}}{}^{*} \frac{1}{1 - \xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}} H_{\mathrm{s+}} \\ + h_{\mathrm{pz}} \frac{1}{1 + \xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}} H_{\mathrm{p-}}{}^{*} + h_{\mathrm{pz}}{}^{*} \frac{1}{1 - \xi_{\mathrm{p}} - i\alpha\xi_{\mathrm{p}}} H_{\mathrm{p+}} \end{array}\right] \tag{17}$$

The sum-frequency response $\mathcal{G}_{2,s+\mathrm{p}}$ is

$$
\begin{aligned}
&(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_{2,\mathrm{s+p}} + \omega_B \mathcal{G}_{2,\mathrm{s+p}} = \left[-\gamma h_z \mathcal{G}_1\right]_{\mathrm{s+p}} \\
&= -\frac{\gamma}{2}\left[\begin{array}{l}\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}}\right)e^{-i\omega_{\mathrm{s+p}}t} \\ +\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}}\right)^{*} e^{+i\omega_{\mathrm{s+p}}t}\end{array}\right] \equiv -\frac{\gamma}{2}\left[S_{2,\mathrm{s+p+}}e^{-i\omega_{\mathrm{s+p}}t} + S_{2,\mathrm{s+p-}}{}^{*}e^{+i\omega_{\mathrm{s+p}}t}\right]
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&(\alpha + i)\frac{\partial}{\partial t}\mathcal{G}_{2,\mathrm{s+p}}{}^{*} + \omega_B \mathcal{G}_{2,\mathrm{s+p}}{}^{*} = \left[-\gamma h_z \mathcal{G}_1{}^{*}\right]_{\mathrm{s+p}} \\
&= -\frac{\gamma}{2}\left[\begin{array}{l}\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}}\right)e^{-i\omega_{\mathrm{s+p}}t} \\ +\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}}\right)^{*} e^{+i\omega_{\mathrm{s+p}}t}\end{array}\right] \equiv -\frac{\gamma}{2}\left[S_{2,\mathrm{s+p-}}e^{-i\omega_{\mathrm{s+p}}t} + S_{2,\mathrm{s+p+}}{}^{*}e^{+i\omega_{\mathrm{s+p}}t}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \mathcal{G}_{2,\mathrm{s+p}} = -\frac{\gamma}{2}\left[\frac{S_{2,\mathrm{s+p+}}}{\omega_B - \omega_{\mathrm{s+p}} - i\alpha\omega_{\mathrm{s+p}}}e^{-i\omega_{\mathrm{s+p}}t} + \frac{S_{2,\mathrm{s+p-}}{}^{*}}{\omega_B + \omega_{\mathrm{s+p}} + i\alpha\omega_{\mathrm{s+p}}}e^{+i\omega_{\mathrm{s+p}}t}\right] \\
&= -\frac{\gamma}{2}\left[\frac{h_{\mathrm{sz}}\Gamma_{\mathrm{p+}}H_{\mathrm{p+}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s+}}H_{\mathrm{s+}}}{\omega_B - \omega_{\mathrm{s+p}} - i\alpha\omega_{\mathrm{s+p}}}e^{-i\omega_{\mathrm{s+p}}t} + \frac{\left(h_{\mathrm{sz}}\Gamma_{\mathrm{p-}}H_{\mathrm{p-}} + h_{\mathrm{pz}}\Gamma_{\mathrm{s-}}H_{\mathrm{s-}}\right)^{*}}{\omega_B + \omega_{\mathrm{s+p}} + i\alpha\omega_{\mathrm{s+p}}}e^{+i\omega_{\mathrm{s+p}}t}\right] \\
&= -\frac{\gamma}{2}\left[\left\{\begin{array}{l}\dfrac{\Gamma_{\mathrm{p+}}}{\omega_B - \omega_{\mathrm{s+p}} - i\alpha\omega_{\mathrm{s+p}}}h_{\mathrm{sz}}H_{\mathrm{p+}} \\ +\dfrac{\Gamma_{\mathrm{s+}}}{\omega_B - \omega_{\mathrm{s+p}} - i\alpha\omega_{\mathrm{s+p}}}h_{\mathrm{pz}}H_{\mathrm{s+}}\end{array}\right\}e^{-i\omega_{\mathrm{s+p}}t} + \left\{\begin{array}{l}\dfrac{\Gamma_{\mathrm{p-}}{}^{*}}{\omega_B + \omega_{\mathrm{s+p}} + i\alpha\omega_{\mathrm{s+p}}}h_{\mathrm{sz}}{}^{*}H_{\mathrm{p-}}{}^{*} \\ +\dfrac{\Gamma_{\mathrm{s-}}{}^{*}}{\omega_B + \omega_{\mathrm{s+p}} + i\alpha\omega_{\mathrm{s+p}}}h_{\mathrm{pz}}{}^{*}H_{\mathrm{s-}}{}^{*}\end{array}\right\}e^{+i\omega_{\mathrm{s+p}}t}\right]
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&= -\left(\frac{\gamma}{2}\right)^2\left[\left\{\begin{array}{l}\dfrac{h_{\mathrm{sz}}H_{\mathrm{p+}}}{\left(\omega_B - \omega_{\mathrm{s+p}} - i\alpha\omega_{\mathrm{s+p}}\right)\left(\omega_B - \omega_{\mathrm{p}} - i\alpha\omega_{\mathrm{p}}\right)} \\ +\dfrac{h_{\mathrm{pz}}H_{\mathrm{s+}}}{\left(\omega_B - \omega_{\mathrm{s+p}} - i\alpha\omega_{\mathrm{s+p}}\right)\left(\omega_B - \omega_{\mathrm{s}} - i\alpha\omega_{\mathrm{s}}\right)}\end{array}\right\}e^{-i\omega_{\mathrm{s+p}}t} + \left\{\begin{array}{l}\dfrac{h_{\mathrm{sz}}{}^{*}H_{\mathrm{p-}}{}^{*}}{\left(\omega_B + \omega_{\mathrm{s+p}} + i\alpha\omega_{\mathrm{s+p}}\right)\left(\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}\right)} \\ +\dfrac{h_{\mathrm{pz}}{}^{*}H_{\mathrm{s-}}{}^{*}}{\left(\omega_B + \omega_{\mathrm{s+p}} + i\alpha\omega_{\mathrm{s+p}}\right)\left(\omega_B + \omega_{\mathrm{s}} + i\alpha\omega_{\mathrm{s}}\right)}\end{array}\right\}e^{+i\omega_{\mathrm{s+p}}t}\right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\left\{\begin{array}{l}\dfrac{h_{\mathrm{sz}}\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)}{\left(1 - \xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1 - \xi_{\mathrm{p}} - i\alpha\xi_{\mathrm{p}}\right)} \\ +\dfrac{h_{\mathrm{pz}}\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right)}{\left(1 - \xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1 - \xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)}\end{array}\right\}e^{-i\omega_{\mathrm{s+p}}t} + \left\{\begin{array}{l}\dfrac{h_{\mathrm{sz}}{}^{*}\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)^{*}}{\left(1 + \xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1 + \xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ +\dfrac{h_{\mathrm{pz}}{}^{*}\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right)^{*}}{\left(1 + \xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1 + \xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)}\end{array}\right\}e^{+i\omega_{\mathrm{s+p}}t}\right]
\end{aligned}
\tag{20}
$$

Finally, the difference-frequency response $\mathcal{G}_{2,s-\mathrm{p}}$ is

$$(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_{2,\mathrm{s-p}} + \omega_B \mathcal{G}_{2,\mathrm{s-p}} = \left[-\gamma \mathrm{h}_z \mathcal{G}_1\right]_{\mathrm{s-p}}$$

$$= -\frac{\gamma}{2}\begin{bmatrix} \left(h_{\mathrm{sz}} \Gamma_{\mathrm{p+}}{}^{*} H_{\mathrm{p+}}{}^{*} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{s-}} H_{\mathrm{s-}} \right) e^{-i\omega_{\mathrm{s-p}} t} \\ + \left(h_{\mathrm{sz}} \Gamma_{\mathrm{p-}}{}^{*} H_{\mathrm{p-}}{}^{*} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{s+}} H_{\mathrm{s+}} \right)^{*} e^{+i\omega_{\mathrm{s-p}} t} \end{bmatrix} \square -\frac{\gamma}{2}\left[S_{2,\mathrm{s-p+}} e^{-i\omega_{\mathrm{s-p}} t} + S_{2,\mathrm{s-p-}}{}^{*} e^{+i\omega_{\mathrm{s-p}} t} \right] \tag{21}$$

$$(\alpha + i)\frac{\partial}{\partial t}\mathcal{G}_{2,\mathrm{s-p}}{}^{*} + \omega_B \mathcal{G}_{2,\mathrm{s-p}}{}^{*} = \left[-\gamma \mathrm{h}_z \mathcal{G}_1{}^{*}\right]_{\mathrm{s-p}}$$

$$= -\frac{\gamma}{2}\begin{bmatrix} \left(h_{\mathrm{sz}} \Gamma_{\mathrm{p-}}{}^{*} H_{\mathrm{p-}}{}^{*} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{s+}} H_{\mathrm{s+}} \right) e^{-i\omega_{\mathrm{s+p}} t} \\ + \left(h_{\mathrm{sz}} \Gamma_{\mathrm{p+}}{}^{*} H_{\mathrm{p+}}{}^{*} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{s-}} H_{\mathrm{s-}} \right)^{*} e^{+i\omega_{\mathrm{s+p}} t} \end{bmatrix} \square -\frac{\gamma}{2}\left[S_{2,\mathrm{s-p-}} e^{-i\omega_{\mathrm{s+p}} t} + S_{2,\mathrm{s-p+}}{}^{*} e^{+i\omega_{\mathrm{s+p}} t} \right]$$

$$\Rightarrow \mathcal{G}_{2,\mathrm{s-p}} = -\frac{\gamma}{2}\left[\frac{S_{2,\mathrm{s-p+}}}{\omega_B - \omega_{\mathrm{s-p}} - i\alpha\omega_{\mathrm{s-p}}} e^{-i\omega_{\mathrm{s-p}} t} + \frac{S_{2,\mathrm{s-p-}}{}^{*}}{\omega_B + \omega_{\mathrm{s-p}} + i\alpha\omega_{\mathrm{s-p}}} e^{+i\omega_{\mathrm{s-p}} t} \right]$$

$$= -\frac{\gamma}{2}\left[\frac{h_{\mathrm{sz}} \Gamma_{\mathrm{p+}}{}^{*} H_{\mathrm{p+}}{}^{*} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{s-}} H_{\mathrm{s-}}}{\omega_B - \omega_{\mathrm{s-p}} - i\alpha\omega_{\mathrm{s-p}}} e^{-i\omega_{\mathrm{s-p}} t} + \frac{\left(h_{\mathrm{sz}} \Gamma_{\mathrm{p-}}{}^{*} H_{\mathrm{p-}}{}^{*} + h_{\mathrm{pz}}{}^{*} \Gamma_{\mathrm{s+}} H_{\mathrm{s+}} \right)^{*}}{\omega_B + \omega_{\mathrm{s-p}} + i\alpha\omega_{\mathrm{s-p}}} e^{+i\omega_{\mathrm{s-p}} t} \right]$$

$$= -\frac{\gamma}{2}\left[\left\{ \begin{matrix} \dfrac{\Gamma_{\mathrm{p+}}{}^{*}}{\omega_B - \omega_{\mathrm{s-p}} - i\alpha\omega_{\mathrm{s-p}}} h_{\mathrm{sz}} H_{\mathrm{p+}}{}^{*} \\ + \dfrac{\Gamma_{\mathrm{s-}}}{\omega_B - \omega_{\mathrm{s-p}} - i\alpha\omega_{\mathrm{s-p}}} h_{\mathrm{pz}}{}^{*} H_{\mathrm{s-}} \end{matrix} \right\} e^{-i\omega_{\mathrm{s-p}} t} + \left\{ \begin{matrix} \dfrac{\Gamma_{\mathrm{p-}}}{\omega_B + \omega_{\mathrm{s-p}} + i\alpha\omega_{\mathrm{s-p}}} h_{\mathrm{sz}}{}^{*} H_{\mathrm{p-}} \\ + \dfrac{\Gamma_{\mathrm{s+}}{}^{*}}{\omega_B + \omega_{\mathrm{s-p}} + i\alpha\omega_{\mathrm{s-p}}} h_{\mathrm{pz}} H_{\mathrm{s+}}{}^{*} \end{matrix} \right\} e^{+i\omega_{\mathrm{s-p}} t} \right]$$

$$= -\left(\frac{\gamma}{2}\right)^2 \left[\left\{ \begin{matrix} \dfrac{h_{\mathrm{sz}} H_{\mathrm{p+}}{}^{*}}{\left(\omega_B - \omega_{\mathrm{s-p}} - i\alpha\omega_{\mathrm{s-p}}\right)\left(\omega_B - \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}{}^{*} H_{\mathrm{s-}}}{\left(\omega_B - \omega_{\mathrm{s-p}} - i\alpha\omega_{\mathrm{s-p}}\right)\left(\omega_B + \omega_{\mathrm{s}} - i\alpha\omega_{\mathrm{s}}\right)} \end{matrix} \right\} e^{-i\omega_{\mathrm{s-p}} t} + \left\{ \begin{matrix} \dfrac{h_{\mathrm{sz}}{}^{*} H_{\mathrm{p-}}}{\left(\omega_B + \omega_{\mathrm{s-p}} + i\alpha\omega_{\mathrm{s-p}}\right)\left(\omega_B + \omega_{\mathrm{p}} + i\alpha\omega_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}} H_{\mathrm{s+}}{}^{*}}{\left(\omega_B + \omega_{\mathrm{s-p}} + i\alpha\omega_{\mathrm{s-p}}\right)\left(\omega_B + \omega_{\mathrm{s}} + i\alpha\omega_{\mathrm{s}}\right)} \end{matrix} \right\} e^{+i\omega_{\mathrm{s-p}} t} \right]$$

$$= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \begin{matrix} \dfrac{h_{\mathrm{sz}}\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)^{*}}{\left(1 - \xi_{\mathrm{s-p}} - i\alpha\xi_{\mathrm{s-p}}\right)\left(1 - \xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}{}^{*}\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right)}{\left(1 - \xi_{\mathrm{s-p}} - i\alpha\xi_{\mathrm{s-p}}\right)\left(1 + \xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)} \end{matrix} \right\} e^{-i\omega_{\mathrm{s-p}} t} + \left\{ \begin{matrix} \dfrac{h_{\mathrm{sz}}{}^{*}\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)}{\left(1 + \xi_{\mathrm{s-p}} + i\alpha\xi_{\mathrm{s-p}}\right)\left(1 + \xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right)^{*}}{\left(1 + \xi_{\mathrm{s-p}} + i\alpha\xi_{\mathrm{s-p}}\right)\left(1 - \xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)} \end{matrix} \right\} e^{+i\omega_{\mathrm{s-p}} t} \right] \tag{22}$$

The various numerators of these terms are evaluated according to our scenario: for the sum-frequency response $\mathcal{G}_{2,s+\mathrm{p}}$,

$$h_{\mathrm{p}x} + ih_{\mathrm{p}y} = \mathcal{H}_{\mathrm{p}x} e^{-i\theta_{\mathrm{p}x}} + i\mathcal{H}_{\mathrm{p}y} e^{-i\theta_{\mathrm{p}y}} \ , \ h_{\mathrm{pz}} = \mathcal{H}_{\mathrm{pz}} e^{-i\theta_{\mathrm{sz}}}$$

$$\mathcal{H}_{\mathrm{p}x} = 0 \ , \ \mathcal{H}_{\mathrm{p}y} = 0 \ , \ \mathcal{H}_{\mathrm{pz}} = \mathcal{H}_{\mathrm{p}c} \ , \ \theta_{\mathrm{pz}} = 0$$

$$h_{\mathrm{s}x} + ih_{\mathrm{s}y} = \mathcal{H}_{\mathrm{s}x} e^{-i\theta_{\mathrm{s}x}} + i\mathcal{H}_{\mathrm{s}y} e^{-i\theta_{\mathrm{s}y}} \ , \ h_{\mathrm{sz}} = \mathcal{H}_{\mathrm{sz}} e^{-i\theta_{\mathrm{sz}}}$$

$$\mathcal{H}_{\mathrm{s}x} = 0 \ , \ \mathcal{H}_{\mathrm{s}y} = \mathcal{H}_{\mathrm{s}c}\cos\varphi \ , \ \mathcal{H}_{\mathrm{sz}} = \mathcal{H}_{\mathrm{s}c}\sin\varphi \ , \ \theta_{y} = 0 \ , \ \theta_{z} = 0$$

$$\Rightarrow \begin{cases} h_{\mathrm{sz}}\left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right) = \mathcal{H}_{\mathrm{s}c}\sin\varphi(0) \\ h_{\mathrm{pz}}\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right) = \mathcal{H}_{\mathrm{p}c}\left(i\mathcal{H}_{\mathrm{s}c}\cos\varphi\right) \\ h_{\mathrm{sz}}^{*}\left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right)^{*} = \mathcal{H}_{\mathrm{s}c}\sin\varphi(0) \\ h_{\mathrm{pz}}^{*}\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right)^{*} = \mathcal{H}_{\mathrm{p}c}\left(i\mathcal{H}_{\mathrm{s}c}\cos\varphi\right) \end{cases} \tag{23}$$

$$\Rightarrow \mathcal{G}_{2,\mathrm{s+p}} = -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \begin{array}{l} \dfrac{(0)\mathcal{H}_{\mathrm{s}c}\sin\varphi}{\left(1-\xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{p}} - i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{\mathcal{H}_{\mathrm{p}c}\left(i\mathcal{H}_{\mathrm{s}c}\cos\varphi\right)}{\left(1-\xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)} \end{array} \right\} e^{-i\omega_{\mathrm{s+p}}t} + \left\{ \begin{array}{l} \dfrac{(0)\mathcal{H}_{\mathrm{s}c}\sin\varphi}{\left(1+\xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{\mathcal{H}_{\mathrm{p}c}\left(i\mathcal{H}_{\mathrm{s}c}\cos\varphi\right)}{\left(1+\xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)} \end{array} \right\} e^{+i\omega_{\mathrm{s+p}}t} \right] \tag{24}$$

$$= -\left(\frac{\gamma}{2\omega_B}\right)^2 \mathcal{H}_{\mathrm{p}c}\mathcal{H}_{\mathrm{s}c}\cos\varphi \left[\left\{ \frac{i}{\left(1-\xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)} \right\} e^{-i\omega_{\mathrm{s+p}}t} + \left\{ \frac{i}{\left(1+\xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)} \right\} e^{+i\omega_{\mathrm{s+p}}t} \right]$$

and for the difference-frequency response $\mathcal{G}_{2,s-\mathrm{p}}$,

$$h_{\mathrm{p}x} + ih_{\mathrm{p}y} = \mathcal{H}_{\mathrm{p}x} e^{-i\theta_{\mathrm{p}x}} + i\mathcal{H}_{\mathrm{p}y} e^{-i\theta_{\mathrm{p}y}} \ , \ h_{\mathrm{pz}} = \mathcal{H}_{\mathrm{pz}} e^{-i\theta_{\mathrm{sz}}}$$

$$\mathcal{H}_{\mathrm{p}x} = 0 \ , \ \mathcal{H}_{\mathrm{p}y} = 0 \ , \ \mathcal{H}_{\mathrm{pz}} = \mathcal{H}_{\mathrm{p}c} \ , \ \theta_{\mathrm{pz}} = 0$$

$$h_{\mathrm{s}x} + ih_{\mathrm{s}y} = \mathcal{H}_{\mathrm{s}x} e^{-i\theta_{\mathrm{s}x}} + i\mathcal{H}_{\mathrm{s}y} e^{-i\theta_{\mathrm{s}y}} \ , \ h_{\mathrm{sz}} = \mathcal{H}_{\mathrm{sz}} e^{-i\theta_{\mathrm{sz}}}$$

$$\mathcal{H}_{\mathrm{s}x} = 0 \ , \ \mathcal{H}_{\mathrm{s}y} = \mathcal{H}_{\mathrm{s}c}\cos\varphi \ , \ \mathcal{H}_{\mathrm{sz}} = \mathcal{H}_{\mathrm{s}c}\sin\varphi \ , \ \theta_{y} = 0 \ , \ \theta_{z} = 0$$

$$\Rightarrow \begin{cases} \left(h_{\mathrm{p}x} + ih_{\mathrm{p}y}\right)^{*} h_{\mathrm{sz}} = (0)\mathcal{H}_{\mathrm{s}c}\sin\varphi \\ h_{\mathrm{pz}}^{*}\left(h_{\mathrm{s}x} - ih_{\mathrm{s}y}\right) = \mathcal{H}_{\mathrm{p}c}\left(-i\mathcal{H}_{\mathrm{s}c}\cos\varphi\right) \\ \left(h_{\mathrm{p}x} - ih_{\mathrm{p}y}\right) h_{\mathrm{sz}}^{*} = (0)\mathcal{H}_{\mathrm{s}c}\sin\varphi \\ h_{\mathrm{pz}}\left(h_{\mathrm{s}x} + ih_{\mathrm{s}y}\right)^{*} = \mathcal{H}_{\mathrm{p}c}\left(-i\mathcal{H}_{\mathrm{s}c}\cos\varphi\right) \end{cases} \tag{25}$$

$$\Rightarrow \mathcal{G}_{2,\text{s-p}} = -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{\begin{array}{l}\dfrac{(0)\mathcal{H}_{sc}\sin\varphi}{(1-\xi_{\text{s-p}}-i\alpha\xi_{\text{s-p}})(1-\xi_{\text{p}}+i\alpha\xi_{\text{p}})} \\ +\dfrac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1-\xi_{\text{s-p}}-i\alpha\xi_{\text{s-p}})(1+\xi_{\text{s}}-i\alpha\xi_{\text{s}})}\end{array}\right\} e^{-i\omega_{\text{s-p}}t} + \left\{\begin{array}{l}\dfrac{(0)\mathcal{H}_{sc}\sin\varphi}{(1+\xi_{\text{s-p}}+i\alpha\xi_{\text{s-p}})(1+\xi_{\text{p}}-i\alpha\xi_{\text{p}})} \\ +\dfrac{\mathcal{H}_{pc}(-i\mathcal{H}_{sc}\cos\varphi)}{(1+\xi_{\text{s-p}}+i\alpha\xi_{\text{s-p}})(1-\xi_{\text{s}}+i\alpha\xi_{\text{s}})}\end{array}\right\} e^{+i\omega_{\text{s-p}}t}\right] \quad (26)$$

$$= -\left(\frac{\gamma}{2\omega_B}\right)^2 \mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi \left[\left\{\frac{-i}{(1-\xi_{\text{s-p}}-i\alpha\xi_{\text{s-p}})(1+\xi_{\text{s}}-i\alpha\xi_{\text{s}})}\right\} e^{-i\omega_{\text{s-p}}t} + \left\{\frac{-i}{(1+\xi_{\text{s-p}}+i\alpha\xi_{\text{s-p}})(1-\xi_{\text{s}}+i\alpha\xi_{\text{s}})}\right\} e^{+i\omega_{\text{s-p}}t}\right]$$

For the z – component terms, we find

$$\text{g}_{z2,\text{s}} = -\frac{1}{2}\mathcal{G}_1^{\,*}\mathcal{G}_1 = -\frac{1}{2}\mathcal{G}_{1,\text{s}}^{\,*}\mathcal{G}_{1,\text{s}} = -\frac{1}{2}\left(\frac{\gamma}{\omega_B}\right)^2 \mathcal{H}_{sc}^{\,2}\cos^2\varphi \left|\frac{i}{1-\xi_{\text{s}}-i\alpha\xi_{\text{s}}}e^{-i\omega_{\text{s}}t} + \frac{i}{1+\xi_{\text{s}}+i\alpha\xi_{\text{s}}}e^{i\omega_{\text{s}}t}\right|^2$$

$$= -\frac{1}{2}\left(\frac{\gamma}{\omega_B}\right)^2 \mathcal{H}_{sc}^{\,2}\cos^2\varphi \left\{\begin{array}{l}\left|\dfrac{1}{(1-\xi_{\text{s}})^2+\alpha^2\xi_{\text{s}}^{\,2}} + \dfrac{1}{(1+\xi_{\text{s}})^2+\alpha^2\xi_{\text{s}}^{\,2}}\right|^2 \\ +\dfrac{1}{1-\xi_{\text{s}}-i\alpha\xi_{\text{s}}}\dfrac{1}{1+\xi_{\text{s}}-i\alpha\xi_{\text{s}}}e^{-2i\omega_{\text{s}}t} + \dfrac{1}{1-\xi_{\text{s}}+i\alpha\xi_{\text{s}}}\dfrac{1}{1+\xi_{\text{s}}+i\alpha\xi_{\text{s}}}e^{2i\omega_{\text{s}}t}\end{array}\right\} \quad (27)$$

$$\text{g}_{z2,\text{p}} = -\frac{1}{2}\mathcal{G}_{1,\text{p}}^{\,*}\mathcal{G}_{1,\text{p}}$$

which do not contain sum or difference contributions.

---

4. Nonlinear Response: Third Order

When the pump field polarization is linear and “vertical,” the third-order equation reads

$$\begin{aligned}(\alpha-i)\frac{\partial}{\partial t}\mathcal{G}_3 + \omega_B\mathcal{G}_3 &= -\gamma \text{h}_z\mathcal{G}_2 \\ (\alpha+i)\frac{\partial}{\partial t}\mathcal{G}_3^{\,*} + \omega_B\mathcal{G}_3^{\,*} &= -\gamma \text{h}_z\mathcal{G}_2^{\,*}\end{aligned} \quad (28)$$

where $\mathcal{G}_2$ is the second-order response

$$
\begin{aligned}
\mathcal{G}_2 &= \mathcal{G}_{2,2\mathrm{s}} + \mathcal{G}_{2,2\mathrm{p}} + \mathcal{G}_{2,\mathrm{s+p}} + \mathcal{G}_{2,\mathrm{s-p}} + \mathcal{G}_{2,0} \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\frac{h_{\mathrm{sz}}\left(h_{\mathrm{s}x} + i h_{\mathrm{s}y}\right)}{\left(1-\xi_{2\mathrm{s}} - i\alpha\xi_{2\mathrm{s}}\right)\left(1-\xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)} e^{-2i\omega_{\mathrm{s}} t} + \frac{h_{\mathrm{sz}}^{*}\left(h_{\mathrm{s}x} - i h_{\mathrm{s}y}\right)^{*}}{\left(1+\xi_{2\mathrm{s}} + i\alpha\xi_{2\mathrm{s}}\right)\left(1+\xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)} e^{+2i\omega_{\mathrm{s}} t} \right] \\
&\quad -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\frac{h_{\mathrm{pz}}\left(h_{\mathrm{p}x} + i h_{\mathrm{p}y}\right)}{\left(1-\xi_{2\mathrm{s}} - i\alpha\xi_{2\mathrm{s}}\right)\left(1-\xi_{\mathrm{p}} - i\alpha\xi_{\mathrm{p}}\right)} e^{-2i\omega_{\mathrm{p}} t} + \frac{h_{\mathrm{pz}}^{*}\left(h_{\mathrm{p}x} - i h_{\mathrm{p}y}\right)^{*}}{\left(1+\xi_{2\mathrm{p}} + i\alpha\xi_{2\mathrm{p}}\right)\left(1+\xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} e^{+2i\omega_{\mathrm{p}} t} \right] \\
&\quad -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \begin{array}{l} \dfrac{h_{\mathrm{sz}}\left(h_{\mathrm{p}x} + i h_{\mathrm{p}y}\right)}{\left(1-\xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{p}} - i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}\left(h_{\mathrm{s}x} + i h_{\mathrm{s}y}\right)}{\left(1-\xi_{\mathrm{s+p}} - i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)} \end{array} \right\} e^{-i\omega_{\mathrm{s+p}} t} + \left\{ \begin{array}{l} \dfrac{h_{\mathrm{sz}}^{*}\left(h_{\mathrm{p}x} - i h_{\mathrm{p}y}\right)^{*}}{\left(1+\xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}^{*}\left(h_{\mathrm{s}x} - i h_{\mathrm{s}y}\right)^{*}}{\left(1+\xi_{\mathrm{s+p}} + i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)} \end{array} \right\} e^{+i\omega_{\mathrm{s+p}} t} \right] \\
&\quad -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\left\{ \begin{array}{l} \dfrac{h_{\mathrm{sz}}\left(h_{\mathrm{p}x} + i h_{\mathrm{p}y}\right)^{*}}{\left(1-\xi_{\mathrm{s-p}} - i\alpha\xi_{\mathrm{s-p}}\right)\left(1-\xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}^{*}\left(h_{\mathrm{s}x} - i h_{\mathrm{s}y}\right)}{\left(1-\xi_{\mathrm{s-p}} - i\alpha\xi_{\mathrm{s-p}}\right)\left(1+\xi_{\mathrm{s}} - i\alpha\xi_{\mathrm{s}}\right)} \end{array} \right\} e^{-i\omega_{\mathrm{s-p}} t} + \left\{ \begin{array}{l} \dfrac{h_{\mathrm{sz}}^{*}\left(h_{\mathrm{p}x} - i h_{\mathrm{p}y}\right)}{\left(1+\xi_{\mathrm{s-p}} + i\alpha\xi_{\mathrm{s-p}}\right)\left(1+\xi_{\mathrm{p}} + i\alpha\xi_{\mathrm{p}}\right)} \\ + \dfrac{h_{\mathrm{pz}}\left(h_{\mathrm{s}x} + i h_{\mathrm{s}y}\right)^{*}}{\left(1+\xi_{\mathrm{s-p}} + i\alpha\xi_{\mathrm{s-p}}\right)\left(1-\xi_{\mathrm{s}} + i\alpha\xi_{\mathrm{s}}\right)} \end{array} \right\} e^{+i\omega_{\mathrm{s-p}} t} \right] \qquad (29) \\
&\quad -\frac{\gamma}{2\omega_B}\left[h_{\mathrm{sz}} \Gamma_{\mathrm{s-}}^{*} H_{\mathrm{s-}}^{*} + h_{\mathrm{pz}} \Gamma_{\mathrm{p-}}^{*} H_{\mathrm{p-}}^{*} + h_{\mathrm{sz}}^{*} \Gamma_{\mathrm{s+}} H_{\mathrm{s+}} + h_{\mathrm{pz}}^{*} \Gamma_{\mathrm{p+}} H_{\mathrm{p+}} \right]
\end{aligned}
$$

Denoting the quantities with dimensionless denominators by $\mathbb{Q}$, we write

$$
\mathcal{G}_2 = -\left(\frac{\gamma}{2\omega_B}\right)^2 \left[\begin{array}{l} \mathbb{Q}_{2\mathrm{s}+} e^{-2i\omega_{\mathrm{s}} t} + \mathbb{Q}_{2\mathrm{s}-} e^{+2i\omega_{\mathrm{s}} t} + \mathbb{Q}_{0} + \mathbb{Q}_{2\mathrm{p}+} e^{-2i\omega_{\mathrm{p}} t} + \mathbb{Q}_{2\mathrm{p}-} e^{+2i\omega_{\mathrm{p}} t} \\ + \mathbb{Q}_{\mathrm{s+p+}} e^{-i\omega_{\mathrm{s+p}} t} + \mathbb{Q}_{\mathrm{s+p-}} e^{+i\omega_{\mathrm{s+p}} t} + \mathbb{Q}_{\mathrm{s-p+}} e^{-i\omega_{\mathrm{s-p}} t} + \mathbb{Q}_{\mathrm{s-p-}} e^{+i\omega_{\mathrm{s-p}} t} \end{array} \right] \qquad (30)
$$

where

$$\square_{2s+}=\frac{h_{sz}\left(h_{sx}+ih_{sy}\right)}{\left(1-\xi_{2s}-i\alpha\xi_{2s}\right)\left(1-\xi_{s}-i\alpha\xi_{s}\right)}\to\frac{\mathcal{H}_{sc}\sin\varphi\left(+i\mathcal{H}_{sc}\cos\varphi\right)}{\left(1-\xi_{2s}-i\alpha\xi_{2s}\right)\left(1-\xi_{s}-i\alpha\xi_{s}\right)}=\frac{i\mathcal{H}_{sc}{}^{2}\sin\varphi\cos\varphi}{\left(1-\xi_{2s}-i\alpha\xi_{2s}\right)\left(1-\xi_{s}-i\alpha\xi_{s}\right)}$$

$$\square_{2s-}=\frac{h_{sz}{}^{*}\left(h_{sx}-ih_{sy}\right)^{*}}{\left(1+\xi_{2s}+i\alpha\xi_{2s}\right)\left(1+\xi_{s}+i\alpha\xi_{s}\right)}\to\frac{\mathcal{H}_{sc}\sin\varphi\left(+i\mathcal{H}_{sc}\cos\varphi\right)}{\left(1+\xi_{2s}+i\alpha\xi_{2s}\right)\left(1+\xi_{s}+i\alpha\xi_{s}\right)}=\frac{i\mathcal{H}_{sc}{}^{2}\sin\varphi\cos\varphi}{\left(1+\xi_{2s}+i\alpha\xi_{2s}\right)\left(1+\xi_{s}+i\alpha\xi_{s}\right)}$$

$$\square_{0}=\frac{h_{sz}\left(h_{sx}-ih_{sy}\right)^{*}}{1+\xi_{s}+i\alpha\xi_{s}}+\frac{h_{pz}\left(h_{px}-ih_{py}\right)^{*}}{1+\xi_{p}+i\alpha\xi_{p}}+\frac{h_{sz}{}^{*}\left(h_{sx}+ih_{sy}\right)}{1-\xi_{s}-i\alpha\xi_{s}}+\frac{h_{pz}{}^{*}\left(h_{px}+ih_{py}\right)}{1-\xi_{p}-i\alpha\xi_{p}}$$

$$\to\frac{\mathcal{H}_{sc}\sin\varphi\left(+i\mathcal{H}_{sc}\cos\varphi\right)}{1+\xi_{s}+i\alpha\xi_{s}}+\frac{\mathcal{H}_{pc}\left(0\right)^{*}}{1+\xi_{p}+i\alpha\xi_{p}}+\frac{\mathcal{H}_{sc}\sin\varphi\left(+i\mathcal{H}_{sc}\cos\varphi\right)}{1-\xi_{s}-i\alpha\xi_{s}}+\frac{\mathcal{H}_{pc}\left(0\right)}{1-\xi_{p}-i\alpha\xi_{p}}$$

$$=\mathcal{H}_{sc}{}^{2}\sin\varphi\cos\varphi\left\{\frac{i}{1+\xi_{s}+i\alpha\xi_{s}}+\frac{i}{1-\xi_{s}-i\alpha\xi_{s}}\right\}$$

$$\square_{2p+}=\frac{h_{pz}\left(h_{px}+ih_{py}\right)}{\left(1-\xi_{2s}-i\alpha\xi_{2s}\right)\left(1-\xi_{p}-i\alpha\xi_{p}\right)}\to\frac{\mathcal{H}_{pc}\left(0\right)}{\left(1-\xi_{2s}-i\alpha\xi_{2s}\right)\left(1-\xi_{p}-i\alpha\xi_{p}\right)}=0$$

$$\square_{2p-}=\frac{h_{pz}{}^{*}\left(h_{px}-ih_{py}\right)^{*}}{\left(1+\xi_{2p}+i\alpha\xi_{2p}\right)\left(1+\xi_{p}+i\alpha\xi_{p}\right)}\to\frac{\mathcal{H}_{pc}\left(0\right)}{\left(1+\xi_{2p}+i\alpha\xi_{2p}\right)\left(1+\xi_{p}+i\alpha\xi_{p}\right)}=0 \qquad (31)$$

$$\square_{s+p+}=\frac{h_{sz}\left(h_{px}+ih_{py}\right)}{\left(1-\xi_{s+p}-i\alpha\xi_{s+p}\right)\left(1-\xi_{p}-i\alpha\xi_{p}\right)}+\frac{h_{pz}\left(h_{sx}+ih_{sy}\right)}{\left(1-\xi_{s+p}-i\alpha\xi_{s+p}\right)\left(1-\xi_{s}-i\alpha\xi_{s}\right)}$$

$$\to\frac{\mathcal{H}_{sc}\sin\varphi\left(0\right)}{\left(1-\xi_{s+p}-i\alpha\xi_{s+p}\right)\left(1-\xi_{p}-i\alpha\xi_{p}\right)}+\frac{\mathcal{H}_{pc}\left(+i\mathcal{H}_{sc}\cos\varphi\right)}{\left(1-\xi_{s+p}-i\alpha\xi_{s+p}\right)\left(1-\xi_{s}-i\alpha\xi_{s}\right)}=\mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi\frac{i}{\left(1-\xi_{s+p}-i\alpha\xi_{s+p}\right)\left(1-\xi_{s}-i\alpha\xi_{s}\right)}$$

$$\square_{s+p-}=\frac{h_{sz}{}^{*}\left(h_{px}-ih_{py}\right)^{*}}{\left(1+\xi_{s+p}+i\alpha\xi_{s+p}\right)\left(1+\xi_{p}+i\alpha\xi_{p}\right)}+\frac{h_{pz}{}^{*}\left(h_{sx}-ih_{sy}\right)^{*}}{\left(1+\xi_{s+p}+i\alpha\xi_{s+p}\right)\left(1+\xi_{s}+i\alpha\xi_{s}\right)}$$

$$\to\frac{\mathcal{H}_{sc}\sin\varphi\left(0\right)}{\left(1+\xi_{s+p}+i\alpha\xi_{s+p}\right)\left(1+\xi_{p}+i\alpha\xi_{p}\right)}+\frac{\mathcal{H}_{pc}\left(+i\mathcal{H}_{sc}\cos\varphi\right)}{\left(1+\xi_{s+p}+i\alpha\xi_{s+p}\right)\left(1+\xi_{s}+i\alpha\xi_{s}\right)}=\mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi\frac{i}{\left(1+\xi_{s+p}+i\alpha\xi_{s+p}\right)\left(1+\xi_{s}+i\alpha\xi_{s}\right)}$$

$$\square_{s-p+}=\frac{h_{sz}\left(h_{px}+ih_{py}\right)^{*}}{\left(1-\xi_{s-p}-i\alpha\xi_{s-p}\right)\left(1-\xi_{p}+i\alpha\xi_{p}\right)}+\frac{h_{pz}{}^{*}\left(h_{sx}-ih_{sy}\right)}{\left(1-\xi_{s-p}-i\alpha\xi_{s-p}\right)\left(1+\xi_{s}-i\alpha\xi_{s}\right)}$$

$$\to\frac{\mathcal{H}_{sc}\sin\varphi\left(0\right)}{\left(1-\xi_{s-p}-i\alpha\xi_{s-p}\right)\left(1-\xi_{p}+i\alpha\xi_{p}\right)}+\frac{\mathcal{H}_{pc}\left(-i\mathcal{H}_{sc}\cos\varphi\right)}{\left(1-\xi_{s-p}-i\alpha\xi_{s-p}\right)\left(1+\xi_{s}-i\alpha\xi_{s}\right)}=\mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi\frac{-i}{\left(1-\xi_{s-p}-i\alpha\xi_{s-p}\right)\left(1+\xi_{s}-i\alpha\xi_{s}\right)} \qquad (32)$$

$$\square_{s-p-}=\frac{h_{sz}{}^{*}\left(h_{px}-ih_{py}\right)}{\left(1+\xi_{s-p}+i\alpha\xi_{s-p}\right)\left(1+\xi_{p}+i\alpha\xi_{p}\right)}+\frac{h_{pz}\left(h_{sx}+ih_{sy}\right)^{*}}{\left(1+\xi_{s-p}+i\alpha\xi_{s-p}\right)\left(1-\xi_{s}+i\alpha\xi_{s}\right)}$$

$$\to\frac{\mathcal{H}_{sc}\sin\varphi\left(0\right)}{\left(1+\xi_{s-p}+i\alpha\xi_{s-p}\right)\left(1+\xi_{p}+i\alpha\xi_{p}\right)}+\frac{\mathcal{H}_{pc}\left(-i\mathcal{H}_{sc}\cos\varphi\right)}{\left(1+\xi_{s-p}+i\alpha\xi_{s-p}\right)\left(1-\xi_{s}+i\alpha\xi_{s}\right)}=\mathcal{H}_{pc}\mathcal{H}_{sc}\cos\varphi\frac{-i}{\left(1+\xi_{s-p}+i\alpha\xi_{s-p}\right)\left(1-\xi_{s}+i\alpha\xi_{s}\right)}$$

We obtain the general source terms from the spectral decomposition of $\mathrm{h}_z\mathcal{G}_2$:

$$\mathrm{h}_z\mathcal{G}_2 = \mathrm{Re}\left\{h_{az}e^{-i\omega_a t} + h_{bz}e^{-i\omega_b t}\right\}\cdot\mathcal{G}_2$$

$$= \frac{1}{2}\begin{pmatrix} h_{\mathrm{sz}}e^{-i\,\omega_{\mathrm{s}}t} \\ +h_{\mathrm{pz}}\,e^{-i\omega_{\mathrm{p}}t} \\ +h_{\mathrm{sz}}{}^{*}e^{+i\,\omega_{\mathrm{s}}t} \\ +h_{\mathrm{pz}}{}^{*}\,e^{+i\omega_{\mathrm{p}}t} \end{pmatrix}\begin{bmatrix} \square_{2s+}e^{-2i\omega_{\mathrm{s}}t} + \square_{2s-}\,e^{+2i\omega_{\mathrm{s}}t} + \square_{0} + \square_{2p+}e^{-2i\omega_{\mathrm{p}}t} + \square_{2p-}\,e^{+2i\omega_{\mathrm{p}}t} \\ +\square_{s+p+}e^{-i\omega_{\mathrm{s+p}}t} + \square_{s+p-}\,e^{+i\omega_{\mathrm{s+p}}t} + \square_{s-p+}e^{-i\omega_{\mathrm{s-p}}t} + \square_{s-p-}\,e^{+i\omega_{\mathrm{s-p}}t} \end{bmatrix} \quad (33)$$

$$h_{\mathrm{sz}}\,e^{-i\omega_{\mathrm{s}}t}\begin{bmatrix} \square_{2s+}e^{-2i\omega_{\mathrm{s}}t} + \square_{2s-}\,e^{+2i\omega_{\mathrm{s}}t} + \square_{0} + \square_{2p+}e^{-2i\omega_{\mathrm{p}}t} + \square_{2p-}\,e^{+2i\omega_{\mathrm{p}}t} \\ +\square_{s+p+}e^{-i\omega_{\mathrm{s+p}}t} + \square_{s+p-}\,e^{+i\omega_{\mathrm{s+p}}t} + \square_{s-p+}e^{-i\omega_{\mathrm{s-p}}t} + \square_{s-p-}\,e^{+i\omega_{\mathrm{s-p}}t} \end{bmatrix}$$

$$= h_{\mathrm{sz}}\square_{2s+}e^{-3i\omega_{\mathrm{s}}t} + h_{\mathrm{sz}}\square_{2s-}\,e^{+i\omega_{\mathrm{s}}t} + h_{\mathrm{sz}}\square_{0}e^{-i\omega_{\mathrm{s}}t} + h_{\mathrm{sz}}\square_{2p+}e^{-i\left(2\omega_{\mathrm{p}} + \omega_{\mathrm{s}}\right)t} + h_{\mathrm{sz}}\square_{2p-}\,e^{+i\left(2\omega_{\mathrm{p}} - \omega_{\mathrm{s}}\right)t}$$

$$+h_{\mathrm{sz}}\square_{s+p+}e^{-i\left(2\omega_{\mathrm{s}} + \omega_{\mathrm{p}}\right)t} + h_{\mathrm{sz}}\square_{s+p-}\,e^{+i\omega_{\mathrm{p}}t} + h_{\mathrm{sz}}\square_{s-p+}e^{-i\left(2\omega_{\mathrm{s}} - \omega_{\mathrm{p}}\right)} + h_{\mathrm{sz}}\square_{s-p-}\,e^{-i\omega_{\mathrm{p}}t}$$

$$h_{\mathrm{pz}}\,e^{-i\omega_{\mathrm{p}}t}\begin{bmatrix} \square_{2s+}e^{-2i\omega_{\mathrm{s}}t} + \square_{2s-}\,e^{+2i\omega_{\mathrm{s}}t} + \square_{0} + \square_{2p+}e^{-2i\omega_{\mathrm{p}}t} + \square_{2p-}\,e^{+2i\omega_{\mathrm{p}}t} \\ +\square_{s+p+}e^{-i\omega_{\mathrm{s+p}}t} + \square_{s+p-}\,e^{+i\omega_{\mathrm{s+p}}t} + \square_{s-p+}e^{-i\omega_{\mathrm{s-p}}t} + \square_{s-p-}\,e^{+i\omega_{\mathrm{s-p}}t} \end{bmatrix}$$

$$= h_{\mathrm{pz}}\square_{2s+}e^{-i\left(2\omega_{\mathrm{s}} + \omega_{\mathrm{p}}\right)t} + h_{\mathrm{pz}}\square_{2s-}e^{+i\left(2\omega_{\mathrm{s}} - \omega_{\mathrm{p}}\right)t} + h_{\mathrm{pz}}\square_{0}\,e^{-i\omega_{\mathrm{p}}t} + h_{\mathrm{pz}}\square_{2p+}e^{-3i\omega_{\mathrm{p}}t} + h_{\mathrm{pz}}\square_{2p-}\,e^{+i\omega_{\mathrm{p}}t}$$

$$+h_{\mathrm{pz}}\square_{s+p+}e^{-i\left(2\omega_{\mathrm{p}} + \omega_{\mathrm{s}}\right)t} + h_{\mathrm{pz}}\square_{s+p-}\,e^{+i\omega_{\mathrm{s}}t} + h_{\mathrm{pz}}\square_{s-p+}e^{-i\omega_{\mathrm{s}}t} + h_{\mathrm{pz}}\square_{s-p-}\,e^{-i\left(2\omega_{\mathrm{p}} - \omega_{\mathrm{s}}\right)t} \quad (34)$$

$$
\begin{aligned}
&h_{sz}^{*}e^{+i\omega_s t}\begin{bmatrix}\square_{2s+}e^{-2i\omega_s t}+\square_{2s-}e^{+2i\omega_s t}+\square_{0}+\square_{2p+}e^{-2i\omega_p t}+\square_{2p-}e^{+2i\omega_p t}\\ +\square_{s+p+}e^{-i\omega_{s+p}t}+\square_{s+p-}e^{+i\omega_{s+p}t}+\square_{s-p+}e^{-i\omega_{s-p}t}+\square_{s-p-}e^{+i\omega_{s-p}t}\end{bmatrix}\\
&=h_{sz}^{*}\square_{2s+}e^{-i\omega_s t}+h_{sz}^{*}\square_{2s-}e^{+3i\omega_s t}+h_{sz}^{*}\square_{0}e^{+i\omega_s t}+h_{sz}^{*}\square_{2p+}e^{-i\left(2\omega_p-\omega_s\right)t}+h_{sz}^{*}\square_{2p-}e^{+i\left(2\omega_p+\omega_s\right)t}\\
&+h_{sz}^{*}\square_{s+p+}e^{-i\omega_p t}+h_{sz}^{*}\square_{s+p-}e^{+i\left(2\omega_s+\omega_p\right)t}+h_{sz}^{*}\square_{s-p+}e^{+i\omega_p t}+h_{sz}^{*}\square_{s-p-}e^{+i\left(2\omega_s-\omega_p\right)t}\\
&h_{pz}^{*}e^{+i\omega_p t}\begin{bmatrix}\square_{2s+}e^{-2i\omega_s t}+\square_{2s-}e^{+2i\omega_s t}+\square_{0}+\square_{2p+}e^{-2i\omega_p t}+\square_{2p-}e^{+2i\omega_p t}\\ +\square_{s+p+}e^{-i\omega_{s+p}t}+\square_{s+p-}e^{+i\omega_{s+p}t}+\square_{s-p+}e^{-i\omega_{s-p}t}+\square_{s-p-}e^{+i\omega_{s-p}t}\end{bmatrix}\\
&=h_{pz}^{*}\square_{2s+}e^{-i\left(2\omega_s-\omega_p\right)t}+h_{pz}^{*}\square_{2s-}e^{+i\left(2\omega_s+\omega_p\right)t}+h_{pz}^{*}\square_{0}e^{+i\omega_p t}+h_{pz}^{*}\square_{2p+}e^{-i\omega_p t}+h_{pz}^{*}\square_{2p-}e^{+3i\omega_p t}\\
&+h_{pz}^{*}\square_{s+p+}e^{-i\omega_s t}+h_{pz}^{*}\square_{s+p-}e^{+i\left(2\omega_p+\omega_s\right)t}+h_{pz}^{*}\square_{s-p+}e^{+i\left(2\omega_p-\omega_s\right)t}+h_{pz}^{*}\square_{s-p-}e^{+i\omega_s t}
\end{aligned}
\tag{35}
$$

For ease of grouping, let us color each term according to its frequency, writing

$$
\begin{aligned}
\mathrm{h}_z\mathcal{G}_2&=h_{sz}\square_{2s+}e^{-3i\omega_s t}+h_{sz}\square_{2s-}e^{+i\omega_s t}+h_{sz}\square_{0}e^{-i\omega_s t}+h_{sz}\square_{2p+}e^{-i\left(2\omega_p+\omega_s\right)t}\\
&+h_{sz}\square_{2p-}e^{+i\left(2\omega_p-\omega_s\right)t}+h_{sz}\square_{s+p+}e^{-i\left(2\omega_s+\omega_p\right)t}+h_{sz}\square_{s+p-}e^{+i\omega_p t}+h_{sz}\square_{s-p+}e^{-i\left(2\omega_s-\omega_p\right)}\\
&+h_{sz}\square_{s-p-}e^{-i\omega_p t}+h_{pz}\square_{2s+}e^{-i\left(2\omega_s+\omega_p\right)t}+h_{pz}\square_{2s-}e^{+i\left(2\omega_s-\omega_p\right)t}+h_{pz}\square_{0}e^{-i\omega_p t}\\
&+h_{pz}\square_{2p+}e^{-3i\omega_p t}+h_{pz}\square_{2p-}e^{+i\omega_p t}+h_{pz}\square_{s+p+}e^{-i\left(2\omega_p+\omega_s\right)t}+h_{pz}\square_{s+p-}e^{+i\omega_s t}\\
&+h_{pz}\square_{s-p+}e^{-i\omega_s t}+h_{pz}\square_{s-p-}e^{-i\left(2\omega_p-\omega_s\right)t}+h_{sz}^{*}\square_{2s+}e^{-i\omega_s t}+h_{sz}^{*}\square_{2s-}e^{+3i\omega_s t}\\
&+h_{sz}^{*}\square_{0}e^{+i\omega_s t}+h_{sz}^{*}\square_{2p+}e^{-i\left(2\omega_p-\omega_s\right)t}+h_{sz}^{*}\square_{2p-}e^{+i\left(2\omega_p+\omega_s\right)t}\\
&+h_{sz}^{*}\square_{s+p+}e^{-i\omega_p t}+h_{sz}^{*}\square_{s+p-}e^{+i\left(2\omega_s+\omega_p\right)t}+h_{sz}^{*}\square_{s-p+}e^{+i\omega_p t}+h_{sz}^{*}\square_{s-p-}e^{+i\left(2\omega_s-\omega_p\right)t}\\
&+h_{pz}^{*}\square_{2s+}e^{-i\left(2\omega_s-\omega_p\right)t}+h_{pz}^{*}\square_{2s-}e^{+i\left(2\omega_s+\omega_p\right)t}+h_{pz}^{*}\square_{0}e^{+i\omega_p t}+h_{pz}^{*}\square_{2p+}e^{-i\omega_p t}+h_{pz}^{*}\square_{2p-}e^{+3i\omega_p t}\\
&+h_{pz}^{*}\square_{s+p+}e^{-i\omega_s t}+h_{pz}^{*}\square_{s+p-}e^{+i\left(2\omega_p+\omega_s\right)t}+h_{pz}^{*}\square_{s-p+}e^{+i\left(2\omega_p-\omega_s\right)t}+h_{pz}^{*}\square_{s-p-}e^{+i\omega_s t}
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&= h_{sz}\square_{2s+} e^{-3i\omega_s t} + h_{sz}{}^{*}\square_{2s-} e^{+3i\omega_s t} \\
&+ h_{sz}\square_{2s-} e^{+i\omega_s t} + h_{sz}\square_{0} e^{-i\omega_s t} + h_{pz}\square_{s+p-} e^{+i\omega_s t} + h_{pz}\square_{s-p+} e^{-i\omega_s t} \\
&+ h_{sz}{}^{*}\square_{2s+} e^{-i\omega_s t} + h_{sz}{}^{*}\square_{0} e^{+i\omega_s t} + h_{pz}{}^{*}\square_{s+p+} e^{-i\omega_s t} + h_{pz}{}^{*}\square_{s-p-} e^{+i\omega_s t} \\
&+ h_{sz}\square_{s+p-} e^{+i\omega_p t} + h_{pz}\square_{2p-} e^{+i\omega_p t} + h_{sz}{}^{*}\square_{s-p+} e^{+i\omega_p t} + h_{pz}{}^{*}\square_{0} e^{+i\omega_p t} \\
&+ h_{sz}\square_{s-p-} e^{-i\omega_p t} + h_{pz}\square_{0} e^{-i\omega_p t} + h_{sz}{}^{*}\square_{s+p+} e^{-i\omega_p t} + h_{pz}{}^{*}\square_{2p+} e^{-i\omega_p t} \\
&+ h_{pz}{}^{*}\square_{s+p-} e^{+i\left(2\omega_p+\omega_s\right)t} + h_{sz}{}^{*}\square_{2p-} e^{+i\left(2\omega_p+\omega_s\right)t} + h_{pz}\square_{s+p+} e^{-i\left(2\omega_p+\omega_s\right)t} + h_{sz}\square_{2p+} e^{-i\left(2\omega_p+\omega_s\right)t} \\
&+ h_{sz}\square_{2p-} e^{+i\left(2\omega_p-\omega_s\right)t} + h_{pz}\square_{s-p-} e^{-i\left(2\omega_p-\omega_s\right)t} + h_{sz}{}^{*}\square_{2p+} e^{-i\left(2\omega_p-\omega_s\right)t} + h_{pz}{}^{*}\square_{s-p+} e^{+i\left(2\omega_p-\omega_s\right)t} \\
&+ h_{sz}\square_{s+p+} e^{-i\left(2\omega_s+\omega_p\right)t} + h_{pz}\square_{2s+} e^{-i\left(2\omega_s+\omega_p\right)t} + h_{sz}{}^{*}\square_{s+p-} e^{+i\left(2\omega_s+\omega_p\right)t} + h_{pz}{}^{*}\square_{2s-} e^{+i\left(2\omega_s+\omega_p\right)t} \\
&+ h_{sz}\square_{s-p+} e^{-i\left(2\omega_s-\omega_p\right)} + h_{sz}{}^{*}\square_{s-p-} e^{+i\left(2\omega_s-\omega_p\right)t} + h_{pz}\square_{2s-} e^{+i\left(2\omega_s-\omega_p\right)t} + + h_{pz}{}^{*}\square_{2s+} e^{-i\left(2\omega_s-\omega_p\right)t} \\
&+ h_{pz}\square_{2p+} e^{-3i\omega_p t} + h_{pz}{}^{*}\square_{2p-} e^{+3i\omega_p t}
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&= h_{sz}\square_{2s+} e^{-3i\omega_s t} + h_{sz}{}^{*}\square_{2s-} e^{+3i\omega_s t} \\
&+ \left(h_{pz}{}^{*}\square_{s-p-} + h_{sz}\square_{2s-} + h_{pz}\square_{s+p-} + h_{sz}{}^{*}\square_{0}\right) e^{+i\omega_s t} + \left(h_{pz}\square_{s-p+} + h_{sz}{}^{*}\square_{2s+} + h_{sz}\square_{0} + h_{pz}{}^{*}\square_{s+p+}\right) e^{-i\omega_s t} \\
&+ \left(h_{sz}\square_{s+p-} + h_{pz}\square_{2p-} + h_{sz}{}^{*}\square_{s-p+} + h_{pz}{}^{*}\square_{0}\right) e^{+i\omega_p t} + \left(h_{sz}\square_{s-p-} + h_{pz}\square_{0} + h_{sz}{}^{*}\square_{s+p+} + h_{pz}{}^{*}\square_{2p+}\right) e^{-i\omega_p t} \\
&+ \left(h_{pz}{}^{*}\square_{s+p-} + h_{sz}{}^{*}\square_{2p-}\right) e^{+i\left(2\omega_p+\omega_s\right)t} + \left(h_{pz}\square_{s+p+} + h_{sz}\square_{2p+}\right) e^{-i\left(2\omega_p+\omega_s\right)t} \\
&+ \left(h_{sz}\square_{2p-} + h_{pz}{}^{*}\square_{s-p+}\right) e^{+i\left(2\omega_p-\omega_s\right)t} + \left(h_{pz}\square_{s-p-} + h_{sz}{}^{*}\square_{2p+}\right) e^{-i\left(2\omega_p-\omega_s\right)t} \\
&+ \left(h_{sz}\square_{s+p+} + h_{pz}\square_{2s+}\right) e^{-i\left(2\omega_s+\omega_p\right)t} + \left(h_{sz}{}^{*}\square_{s+p-} + h_{pz}{}^{*}\square_{2s-}\right) e^{+i\left(2\omega_s+\omega_p\right)t} \\
&+ \left(h_{sz}\square_{s-p+} + h_{pz}{}^{*}\square_{2s+}\right) e^{-i\left(2\omega_s-\omega_p\right)} + \left(h_{sz}{}^{*}\square_{s-p-} + h_{pz}\square_{2s-}\right) e^{+i\left(2\omega_s-\omega_p\right)t} \\
&+ h_{pz}\square_{2p+} e^{-3i\omega_p t} + h_{pz}{}^{*}\square_{2p-} e^{+3i\omega_p t}
\end{aligned}
\tag{38}
$$

Then, the spectral decomposition can finally be written

$$
\begin{aligned}
&\mathrm{h}_z \mathcal{G}_2 = h_{\mathrm{sz}} \square_{2s+} e^{-3i\omega_{\mathrm{s}} t} + h_{\mathrm{sz}}^{*} \square_{2s-} e^{+3i\omega_{\mathrm{s}} t} \\
&+\left(h_{\mathrm{pz}} \square_{s+p-} + h_{\mathrm{sz}} \square_{2s-} + h_{\mathrm{pz}}^{*} \square_{s-p-} + h_{\mathrm{sz}}^{*} \square_{0} \right) e^{+i\omega_{\mathrm{s}} t} + \left(h_{\mathrm{sz}} \square_{0} + h_{\mathrm{pz}} \square_{s-p+} + h_{\mathrm{pz}}^{*} \square_{s+p+} + h_{\mathrm{sz}}^{*} \square_{2s+} \right) e^{-i\omega_{\mathrm{s}} t} \\
&+\left(h_{\mathrm{sz}} \square_{s+p-} + h_{\mathrm{pz}} \square_{2p-} + h_{\mathrm{sz}}^{*} \square_{s-p+} + h_{\mathrm{pz}}^{*} \square_{0} \right) e^{+i\omega_{\mathrm{p}} t} + \left(h_{\mathrm{pz}} \square_{0} + h_{\mathrm{sz}} \square_{s-p-} + h_{\mathrm{sz}}^{*} \square_{s+p+} + h_{\mathrm{pz}}^{*} \square_{2p+} \right) e^{-i\omega_{\mathrm{p}} t} \\
&+\left(h_{\mathrm{pz}}^{*} \square_{s+p-} + h_{\mathrm{sz}}^{*} \square_{2p-} \right) e^{+i\left(2\omega_{\mathrm{p}} + \omega_{\mathrm{s}}\right)t} + \left(h_{\mathrm{pz}} \square_{s+p+} + h_{\mathrm{sz}} \square_{2p+} \right) e^{-i\left(2\omega_{\mathrm{p}} + \omega_{\mathrm{s}}\right)t} \\
&+\left(h_{\mathrm{pz}}^{*} \square_{s-p+} + h_{\mathrm{sz}} \square_{2p-} \right) e^{+i\left(2\omega_{\mathrm{p}} - \omega_{\mathrm{s}}\right)t} + \left(h_{\mathrm{pz}} \square_{s-p-} + h_{\mathrm{sz}}^{*} \square_{2p+} \right) e^{-i\left(2\omega_{\mathrm{p}} - \omega_{\mathrm{s}}\right)t} \\
&+\left(h_{\mathrm{sz}}^{*} \square_{s+p-} + h_{\mathrm{pz}}^{*} \square_{2s-} \right) e^{+i\left(2\omega_{\mathrm{s}} + \omega_{\mathrm{p}}\right)t} + \left(h_{\mathrm{sz}} \square_{s+p+} + h_{\mathrm{pz}} \square_{2s+} \right) e^{-i\left(2\omega_{\mathrm{s}} + \omega_{\mathrm{p}}\right)t} \\
&+\left(h_{\mathrm{sz}}^{*} \square_{s-p-} + h_{\mathrm{pz}} \square_{2s-} \right) e^{+i\left(2\omega_{\mathrm{s}} - \omega_{\mathrm{p}}\right)t} + \left(h_{\mathrm{sz}} \square_{s-p+} + h_{\mathrm{pz}}^{*} \square_{2s+} \right) e^{-i\left(2\omega_{\mathrm{s}} - \omega_{\mathrm{p}}\right)} \\
&+h_{\mathrm{pz}} \square_{2p+} e^{-3i\omega_{\mathrm{p}} t} + h_{\mathrm{pz}}^{*} \square_{2p-} e^{+3i\omega_{\mathrm{p}} t}
\end{aligned}
\tag{39}
$$

Because we are interested only in the components that arise from mixing of the pump and signal, we solve for the sideband (intermod) responses only:

$$
(\alpha - i)\frac{\partial}{\partial t}\mathcal{G}_3\Big|_{\mathrm{sideband}} + \omega_B \mathcal{G}_3\Big|_{\mathrm{sideband}} = \frac{\gamma}{2}\left\{
\begin{aligned}
&S_{3,\mathrm{sp2p},+} e^{-i\left(\omega_{\mathrm{s}} + 2\omega_{\mathrm{p}}\right)t} + S_{3,\mathrm{sp2p},-} e^{+i\left(\omega_{\mathrm{s}} + 2\omega_{\mathrm{p}}\right)t} \\
&+S_{3,\mathrm{sm2p},+} e^{-i\left(\omega_{\mathrm{s}} - 2\omega_{\mathrm{p}}\right)t} + S_{3,\mathrm{sm2p},-} e^{+i\left(\omega_{\mathrm{s}} - 2\omega_{\mathrm{p}}\right)t}
\end{aligned}
\right\}
\tag{40}
$$

where

$$
\begin{aligned}
S_{3,\mathrm{s+2p},+} &= h_{\mathrm{pz}} \square_{s+p+} + h_{\mathrm{sz}} \square_{2p+} = h_{\mathrm{pz}} \square_{s+p+} \\
S_{3,\mathrm{s+2p},-} &= h_{\mathrm{pz}}^{*} \square_{s+p-} + h_{\mathrm{sz}}^{*} \square_{2p-} = h_{\mathrm{pz}}^{*} \square_{s+p-} \\
S_{3,\mathrm{s-2p},+} &= h_{\mathrm{pz}}^{*} \square_{\mathrm{s-p+}} + h_{\mathrm{sz}} \square_{2p-} = h_{\mathrm{pz}}^{*} \square_{\mathrm{s-p+}} \\
S_{3,\mathrm{s-2p},-} &= h_{\mathrm{pz}} \square_{\mathrm{s-p-}} + h_{\mathrm{sz}}^{*} \square_{2p+} = h_{\mathrm{pz}} \square_{\mathrm{s-p-}}
\end{aligned}
\tag{41}
$$

Let $\mathcal{G}_3\Big|_{\mathrm{sideband}} = \mathcal{G}_{3,\mathrm{sp2p},+} + \mathcal{G}_{3,\mathrm{sp2p},-} + \mathcal{G}_{3,\mathrm{sm2p},+} + \mathcal{G}_{3,\mathrm{sm2p},-}$. Then for the upper sideband, we have

$$
\begin{aligned}
\mathcal{G}_{2,\mathrm{s+p}} &= -\frac{\gamma}{2}\left[\frac{S_{2,\mathrm{s+p},+}}{\omega_B-\omega_{\mathrm{s+p}}-i\alpha\omega_{\mathrm{s+p}}}e^{-i\omega_{\mathrm{s+p}}t}+\frac{S_{2,\mathrm{s+p},-}{}^{*}}{\omega_B+\omega_{\mathrm{s+p}}+i\alpha\omega_{\mathrm{s+p}}}e^{+i\omega_{\mathrm{s+p}}t}\right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\left\{\begin{array}{l}\dfrac{h_{\mathrm{sz}}\left(h_{\mathrm{p}x}+ih_{\mathrm{p}y}\right)}{\left(1-\xi_{\mathrm{s+p}}-i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{p}}-i\alpha\xi_{\mathrm{p}}\right)} \\ +\dfrac{h_{\mathrm{pz}}\left(h_{\mathrm{s}x}+ih_{\mathrm{s}y}\right)}{\left(1-\xi_{\mathrm{s+p}}-i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{s}}-i\alpha\xi_{\mathrm{s}}\right)}\end{array}\right\}e^{-i\omega_{\mathrm{s+p}}t}+\left\{\begin{array}{l}\dfrac{h_{\mathrm{sz}}{}^{*}\left(h_{\mathrm{p}x}-ih_{\mathrm{p}y}\right)^{*}}{\left(1+\xi_{\mathrm{s+p}}+i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{p}}+i\alpha\xi_{\mathrm{p}}\right)} \\ +\dfrac{h_{\mathrm{pz}}{}^{*}\left(h_{\mathrm{s}x}-ih_{\mathrm{s}y}\right)^{*}}{\left(1+\xi_{\mathrm{s+p}}+i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{s}}+i\alpha\xi_{\mathrm{s}}\right)}\end{array}\right\}e^{+i\omega_{\mathrm{s+p}}t}\right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\square_{\mathrm{s+p}+}e^{-i\omega_{\mathrm{s+p}}t}+\square_{\mathrm{s+p}-}e^{+i\omega_{\mathrm{s+p}}t}\right] \\
\Rightarrow \mathcal{G}_{3,\mathrm{s+2p}} &= -\frac{\gamma}{2}\left[\frac{S_{3,\mathrm{s+2p},+}}{\omega_B-\omega_{\mathrm{s+2p}}-i\alpha\omega_{\mathrm{s+2p}}}e^{-i\omega_{\mathrm{s+2p}}t}+\frac{S_{3,\mathrm{s+2p},-}{}^{*}}{\omega_B+\omega_{\mathrm{s+2p}}+i\alpha\omega_{\mathrm{s+2p}}}e^{+i\omega_{\mathrm{s+2p}}t}\right] \\
&= -\frac{\gamma}{2}\left(\frac{\gamma}{2\omega_B}\right)^2\left[\frac{h_{\mathrm{pz}}}{\omega_B-\omega_{\mathrm{s+2p}}-i\alpha\omega_{\mathrm{s+2p}}}\square_{\mathrm{s+p}+}e^{-i\omega_{\mathrm{s+2p}}t}+\frac{h_{\mathrm{pz}}{}^{*}}{\omega_B+\omega_{\mathrm{s+2p}}+i\alpha\omega_{\mathrm{s+2p}}}\square_{\mathrm{s+p}-}e^{+i\omega_{\mathrm{s+2p}}t}\right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^3\mathcal{H}_{\mathrm{p}c}{}^2\mathcal{H}_{\mathrm{s}c}\cos\varphi\left[\begin{array}{l}\dfrac{i}{\left(1-\xi_{\mathrm{s+2p}}-i\alpha\xi_{\mathrm{s+2p}}\right)\left(1-\xi_{\mathrm{s+p}}-i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{s}}-i\alpha\xi_{\mathrm{s}}\right)}e^{-i\omega_{\mathrm{s+2p}}t} \\ +\dfrac{i}{\left(1+\xi_{\mathrm{s+2p}}+i\alpha\xi_{\mathrm{s+2p}}\right)\left(1+\xi_{\mathrm{s+p}}+i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{s}}+i\alpha\xi_{\mathrm{s}}\right)}e^{+i\omega_{\mathrm{s+2p}}t}\end{array}\right] \\
&= -\left(\frac{\gamma}{2\omega_B}\right)^3\mathcal{H}_{\mathrm{p}c}{}^2\mathcal{H}_{\mathrm{s}c}\cos\varphi\left[\begin{array}{l}\dfrac{i}{\left(1-\xi_{\mathrm{s+2p}}-i\alpha\xi_{\mathrm{s+2p}}\right)\left(1-\xi_{\mathrm{s+p}}-i\alpha\xi_{\mathrm{s+p}}\right)\left(1-\xi_{\mathrm{s}}-i\alpha\xi_{\mathrm{s}}\right)}e^{-i\omega_{\mathrm{s+2p}}t} \\ +\dfrac{i}{\left(1+\xi_{\mathrm{s+2p}}+i\alpha\xi_{\mathrm{s+2p}}\right)\left(1+\xi_{\mathrm{s+p}}+i\alpha\xi_{\mathrm{s+p}}\right)\left(1+\xi_{\mathrm{s}}+i\alpha\xi_{\mathrm{s}}\right)}e^{+i\omega_{\mathrm{s+2p}}t}\end{array}\right] \qquad (42)
\end{aligned}
$$

Likewise, for the lower sideband, we have

$$
\begin{aligned}
\mathcal{G}_{2,\mathrm{s-p}} &= -\frac{\gamma}{2}\left[\frac{S_{2,\mathrm{s-p},+}}{\omega_B-\omega_{\mathrm{s-p}}-i\alpha\omega_{\mathrm{s-p}}}e^{-i\omega_{\mathrm{s-p}}t}+\frac{S_{2,\mathrm{s-p},-}^{\;*}}{\omega_B+\omega_{\mathrm{s-p}}+i\alpha\omega_{\mathrm{s-p}}}e^{+i\omega_{\mathrm{s-p}}t}\right]\\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\left\{\begin{array}{l}\dfrac{h_{\mathrm{s}z}\left(h_{\mathrm{p}x}+ih_{\mathrm{p}y}\right)^*}{\left(1-\xi_{\mathrm{s-p}}-i\alpha\xi_{\mathrm{s-p}}\right)\left(1-\xi_{\mathrm{p}}+i\alpha\xi_{\mathrm{p}}\right)}\\ +\dfrac{h_{\mathrm{p}z}^{\;*}\left(h_{\mathrm{s}x}-ih_{\mathrm{s}y}\right)}{\left(1-\xi_{\mathrm{s-p}}-i\alpha\xi_{\mathrm{s-p}}\right)\left(1+\xi_{\mathrm{s}}-i\alpha\xi_{\mathrm{s}}\right)}\end{array}\right\}e^{-i\omega_{\mathrm{s-p}}t}+\left\{\begin{array}{l}\dfrac{h_{\mathrm{s}z}^{\;*}\left(h_{\mathrm{p}x}-ih_{\mathrm{p}y}\right)}{\left(1+\xi_{\mathrm{s-p}}+i\alpha\xi_{\mathrm{s-p}}\right)\left(1+\xi_{\mathrm{p}}+i\alpha\xi_{\mathrm{p}}\right)}\\ +\dfrac{h_{\mathrm{p}z}\left(h_{\mathrm{s}x}+ih_{\mathrm{s}y}\right)^*}{\left(1+\xi_{\mathrm{s-p}}+i\alpha\xi_{\mathrm{s-p}}\right)\left(1-\xi_{\mathrm{s}}+i\alpha\xi_{\mathrm{s}}\right)}\end{array}\right\}e^{+i\omega_{\mathrm{s-p}}t}\right]\\
&= -\left(\frac{\gamma}{2\omega_B}\right)^2\left[\square_{\mathrm{s-p}+}e^{-i\omega_{\mathrm{s-p}}t}+\square_{\mathrm{s-p}-}e^{+i\omega_{\mathrm{s-p}}t}\right]\\
\Rightarrow \mathcal{G}_{2,\mathrm{s-2p}} &= -\frac{\gamma}{2}\left[\frac{S_{3,\mathrm{s-2p},+}}{\omega_B-\omega_{\mathrm{s-2p}}-i\alpha\omega_{\mathrm{s-2p}}}e^{-i\omega_{\mathrm{s-2p}}t}+\frac{S_{3,\mathrm{s-2p},-}^{\;*}}{\omega_B+\omega_{\mathrm{s-2p}}+i\alpha\omega_{\mathrm{s-2p}}}e^{+i\omega_{\mathrm{s-2p}}t}\right]\\
&= -\frac{\gamma}{2}\left(\frac{\gamma}{2\omega_B}\right)^2\left[\frac{h_{\mathrm{p}z}^{\;*}\square_{\mathrm{s-p}+}}{\omega_B-\omega_{\mathrm{s-2p}}-i\alpha\omega_{\mathrm{s-2p}}}e^{-i\omega_{\mathrm{s-2p}}t}+\frac{h_{\mathrm{p}z}\square_{\mathrm{s-p}-}}{\omega_B+\omega_{\mathrm{s-2p}}+i\alpha\omega_{\mathrm{s-2p}}}e^{+i\omega_{\mathrm{s-2p}}t}\right]\\
&= -\left(\frac{\gamma}{2\omega_B}\right)^3\mathcal{H}_{\mathrm{p}c}^{\;2}\mathcal{H}_{\mathrm{s}c}\cos\varphi\left[\begin{array}{l}\dfrac{-i}{\left(1-\xi_{\mathrm{s-2p}}-i\alpha\xi_{\mathrm{s-2p}}\right)\left(1-\xi_{\mathrm{s-p}}-i\alpha\xi_{\mathrm{s-p}}\right)\left(1+\xi_{\mathrm{s}}-i\alpha\xi_{\mathrm{s}}\right)}e^{-i\omega_{\mathrm{s-2p}}t}\\ +\dfrac{-i}{\left(1+\xi_{\mathrm{s-2p}}+i\alpha\xi_{\mathrm{s-2p}}\right)\left(1+\xi_{\mathrm{s-p}}+i\alpha\xi_{\mathrm{s-p}}\right)\left(1-\xi_{\mathrm{s}}+i\alpha\xi_{\mathrm{s}}\right)}e^{+i\omega_{\mathrm{s-2p}}t}\end{array}\right]
\end{aligned}
\tag{43}
$$

These expressions are the ones used in reference 1.

5. Conclusion

The complexity of these results, even for the idealized geometry chosen here, makes them problematic for use in deciphering experimental data from, e.g., pump illumination by a moving airframe. Nevertheless, the effort could be worthwhile if the signal returns were strong enough. Unfortunately, as reference 1 shows, this requires very high levels of pump power even over short distances.

6. References

1. Crowne, F. J. Multifrequency Excitation of a Ferromagnetic Microwire. *IEEE Trans. Magnetics*, to be published.

2. Douse, W.; King, W. unpublished.

3. Crowne, F. J. *Perturbation Theory for the Landau-Lifshits-Gilbert Equation*; ARL-TR-6114; U.S. Army Research Laboratory: Adelphi, MD, September 2012.

4. Landau, L. D.; Lifshits, E. M. Theory of the Dispersion of Magnetic Permeability in Ferromagnetic Bodies. *Phys. Z. Sowietunion* **1935**, *8*, 153.

5. Kittel, C. *Phys. Rev.* **1948**, *73*, 155.

6. Gilbert, T. L. A Phenomenological Theory of Damping in Ferromagnetic Materials. *IEEE Trans. Magnetics* **2004**, *40*, 3443.

NO. OF COPIES	ORGANIZATION
1 ELEC	ADMNSTR DEFNS TECHL INFO CTR ATTN DTIC OCP 8725 JOHN J KINGMAN RD STE 0944 FT BELVOIR VA 22060-6218
1	US ARMY RSRCH DEV AND ENGRG CMND ARMAMENT RSRCH DEV & ENGRG CTR ARMAMENT ENGRG & TECHNLGY CTR ATTN AMSRD AAR AEF T J MATTS BLDG 305 ABERDEEN PROVING GROUND MD 21005-5001
1	US ARMY ARDEC ATTN AMSRD AAR AEE P C HAINES BLDG 25 PICATINNY ARSENAL NJ 07806-5000
1	US ARMY ARDEC ATTN AMSRD AAR AEM C A MARSTON BLDG 61S PICATINNY ARSENAL NJ 07806-5000
2	US ARMY ARDEC ATTN AMSRD AAR AEM L S GILMAN ATTN AMSRD AAR AEP F O NGUYEN BLDG 65S PICATINNY ARSENAL NJ 07806-5000

NO. OF COPIES	ORGANIZATION
1	US ARMY INFO SYS ENGRG CMND ATTN AMSEL IE TD A RIVERA FT HUACHUCA AZ 85613-5300
1	US GOVERNMENT PRINT OFF DEPOSITORY RECEIVING SECTION ATTN MAIL STOP IDAD J TATE 732 NORTH CAPITOL ST NW WASHINGTON DC 20402
26	US ARMY RSRCH LAB ATTN IMAL HRA MAIL & RECORDS MGMT ATTN RDRL CI J PELLEGRINO ATTN RDRL CIO LL TECHL LIB ATTN RDRL CIO LT TECHL PUB ATTN RDRL SEE I S SVENSSON ATTN RDRL SEE O W M GOLDING ATTN RDRL SER E B HUEBSCHMAN ATTN RDRL SER E F CROWNE (5 COPIES) ATTN RDRL SER E E VIVEIROS ATTN RDRL SER E G BIRDWELL ATTN RDRL SER E T O’REGAN ATTN RDRL SER E P SHAH ATTN RDRL SER E R DEL ROSARIO ATTN RDRL SER E T IVANOV ATTN RDRL SER J MAIT ATTN RDRL SER L J PULSKAMP ATTN RDRL SER L L CURRANO ATTN RDRL SER L M CHIN ATTN RDRL SER L M ERVIN ATTN RDRL SER L R POLCAWICH ATTN RDRL SER P AMIRTHARAJ ATTN RDRL SER U C FAZI ADELPHI MD 20783-1197